Setting Up and Solving Optimization Problems with Calculus

Consider the following problem:

> A landscape architect plans to enclose a 3000 square foot rectangular region in a botanical garden. She will use shrubs costing \$25 per foot along three sides and fencing costing \$10 per foot along the fourth side. What dimensions of the rectangular region will result in the least possible cost?

Problems like this, which ask us to determine certain values in order to either maximize or minimize a certain quantity, are called *optimization problems*. These are an extremely important class of problems, but can be challenging because they often require multiple steps to solve. Understanding these steps will help you tackle even complicated optimization problems.

The left-hand column below lists the general steps in the order they are typically done. The right-hand column shows how each step is applied to our particular problem.

Procedure

1. Read the problem, then read it again. Begin by reading the problem. It's okay if not everything makes sense at first, but try and get an idea of what the problem about and what it's asking you to do. What types of quantities are involved? What are their units?

Solution

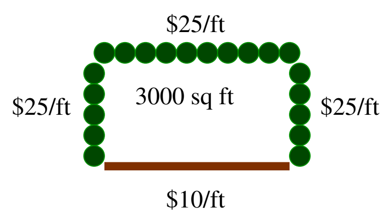
Problem will involve *distance* (feet), *area* (sq. ft), and *cost* (dollars).

Procedure

2. Start writing things down and, in most cases, **draw a picture**. Often you can start with a sketch. Don't worry about artistic quality; the point of the diagram is to help you visualize the problem, provide a place for you to label unknown quantities, and ultimately help you translate the problem into a mathematical expression.

Solution

Here is a top-down view of the garden. The three sides with shrubs cost \$25/ft while the fourth side costs \$10/ft.

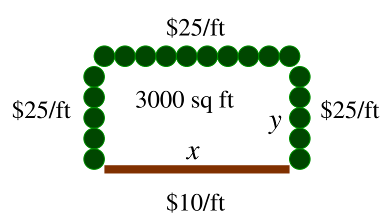


Procedure

3. Name and label unknown quantities. Make up appropriate variable names for any quantities you don't know and think you might need. Put these on your diagram.

Solution

Let x be the width and y be the height of the rectangle. The names used here are somewhat arbitrary. We could have just as easily used l and w for length and width.



Procedure (continued)

4. Determine what you're trying to optimize. Clearly identify the *objective*, which is the quantity you want to maximize or minimize.

5. Develop an equation with the objective. Develop an equation that relates the objective to other known and unknown (but named) quantities. Use your diagram and pay close attention to units.

6. If necessary, rewrite the objective in terms of a single variable. Use algebra and additional relationships to rewrite the objective equation so it expresses the objective as a function of a single variable.

7. Determine the domain. Clearly identify the set of values the independent variable can take on. The endpoints of this domain (if they exist) may correspond to extrema.

Solution (continued)

In this problem we want to minimize cost. Let's call this C. Note that C will depend on the total length of the shrub border, i.e. the perimeter of the garden, and so will depend on both x and y.

The total cost for the shrubs is given by

$$C = 25y + 25x + 25y + 10x$$

so $C = 35x + 50y$.

Our objective is a function of both x and y. We can eliminate one of these by expressing it in terms of the other and then replacing it in the objective. Since the area of the garden is 3000 sq. ft., we know $3000 = xy$. Solving for y we have

$$y = \frac{3000}{x}.$$

Replacing y in our objective, we obtain

$$C = 35x + 50\left(\frac{3000}{x}\right) = 35x + \frac{150000}{x}.$$

Notice that $x \neq 0$ since otherwise C is undefined. It also makes sense that $x > 0$ since x represents a length. In this problem there is no upper-bound on how large x can be, so the domain of our objective is the open interval $(0, \infty)$.

Procedure (continued)

8. Find all extrema. Finally we get to use calculus! Usually this means finding the absolute extreme values of the objective. Remember, this means we need to identify and check all critical numbers as well as any endpoints of our domain.

9. Review the problem statement. Pay special attention to the form of the question or problem. Does your answer make sense? Are the units correct?

10. State your answer. Okay, you've done the math, now you need to communicate your answer clearly. Provide an answer for the problem using a **complete sentence**. *Note: The variable names you introduced when solving your problem should not appear in your answer!*

Solution (continued)

Computing the derivative of C with respect to x we have

$$C'(x) = 35 - \frac{150000}{x^2}.$$

This is defined for all $x \neq 0$ so C' is defined everywhere that C is. Thus all possible critical number(s) can be found by setting $C'(x) = 0$. This gives

$$35 - \frac{150000}{x^2} = 0$$

and solving for x gives

$$x = \sqrt{\frac{150000}{35}} \approx 65.4654$$

as the only other critical number, so $C(\sqrt{150000/35}) \approx 4582.5757$ is the absolute minimum cost.

Our problem statement asked for the dimensions of the rectangle enclosing 3000 sq. ft. that has the least expensive border of shrubs. We already have x, the garden's width along the side bordered by the less expensive shrubs, and just need to compute $y = 3000/x \approx 45.8258$ to have the garden's height. It makes sense that the cheaper shrubs are along the longer side.

Answer: In order to minimize the cost, the garden should be approximately 65.47 ft $\times$ 45.83 ft, with the \$10 shrubs planted along one of the longer sides. The total cost of the shrubs will be \$4,582.58.

Note: Not all of the steps will be required to solve every optimization problem.

Problems

1. Suppose you have 30 feet of fencing and want to fence in a rectangular garden next to a house. You want the largest possible garden, so you decide to use one wall of the house as a border and use the fencing for the other three sides. What should the dimensions of the rectangle be?

2. A square-bottomed box with no top has a fixed volume V. What dimensions minimize the surface area?