Antiderivatives

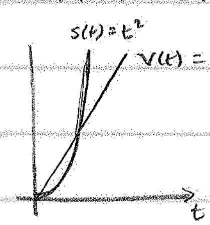
The derivative of $f(x)$ gives us the rate of change of f with respect to x at x.

What would the inverse operation give us? Think of the position function $s(t)$. $s'(t)$ is the velocity. Now suppose we only knew the velocity function $v(t) = s'(t)$. How can we recover position?

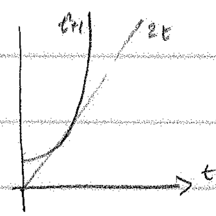
Key idea: derivatives $\longrightarrow$ rates of change.
antiderivatives $\longrightarrow$ <u>cumulative</u> effect of changes.

Ex $\quad v(t) = 2t$

What is $s(t)$ so $s'(t) = 2t$? $\boxed{s(t) = t^2}$



What about $s(t) = t^2 + 1$?
$s'(t) = 2t$



So this function also works

Notice that when we "undid" the derivative of position, we found the total change in position <u>relative to some arbitrary starting point</u>.

In general $s(t) = t^2 + C$ for any constant C.

Antiderivatives

The process of "undoing" a derivative is called antidifferentiation.

Thm. Suppose that F and G are both antiderivatives of f on the interval I. Then

$$F(x) = G(x) + c$$

for some constant c.

Proof Since $F'(x) = f = G'(x)$ we know $F'(x) = G'(x)$

Let $h(x) = F(x) - G(x)$ then $h'(x) = F'(x) - G'(x) = 0$

Since $h'(x) = 0$ we know $h(x) = c$. therefore

$$c = F(x) - G(x) \quad \text{or} \quad F(x) = G(x) + c \checkmark$$

Definition Let F be any antiderivative of f. The indefinite integral of $f(x)$ (with respect to x) is defined by

$$\int f(x)\,dx = F(x) + c$$

where c is an arbitray constant called the constant of integration.

Ex

$$\int 2x\,dx = x^2 + c \qquad \int \frac{1}{2\sqrt{x}}\,dx = \sqrt{x} + c$$

$$\int \cos x\,dx = \sin x + c \qquad \int x\,dx = \frac{1}{2}x^2 + c$$

$$\int dx = x + c \qquad \int x^3\,dx = \frac{1}{4}x^4 + c$$

Antiderivatives

Thm (Power Rule)

For any rational power $r \neq -1$ $\quad \int x^r \, dx = \frac{x^{r+1}}{r+1} + c$

Ex $\int x^5 \, dx = \frac{x^6}{6} + c$

Ex $\int \sqrt{x} \, dx = \frac{x^{3/2}}{3/2} + c = \frac{2}{3} x^{3/2} + c = \frac{2}{3}\sqrt{x^3} + c$

Ex $\int \frac{1}{x^3} dx = \int x^{-3} dx = \frac{x^{-2}}{-2} dx = -\frac{1}{2x^2} + c$

We can easily construct other antiderivative rules from differentiation formulas we know already.

Since $\frac{d}{dx} \sin^{-1} x = \frac{1}{\sqrt{1-x^2}}$, we know $\int \frac{1}{\sqrt{1-x^2}} dx = \sin^{-1} x + c$

Ex $\int \sec x \tan x \, dx = \sec x + c$

Ex $\int e^x dx = e^x + c$

Ex $\int e^{-x} dx = -e^{-x} + c$

Thm. Like differentiation, antidifferentiation is a linear operation. This means that, for functions f and g that have antiderivatives and for any constants a and b

$$\int [af(x) + bg(x)] \, dx = a \int f(x) \, dx + b \int g(x) \, dx$$

Ex $\int 3x^2+2\sin x \, dx = \int 3x^2 dx + 2\int \sin x \, dx$

$= x^3 + 2(-\cos x) + c$

$= x^3 - 2\cos x + c$

How can we compute $\int \frac{1}{x} dx$?

Clearly $x \neq 0$. When $x > 0$ we recall that $\frac{d}{dx} \ln x = \frac{1}{x}$

So we have

$$\int \frac{1}{x} dx = \ln x + c \quad \text{when } x > 0$$

What if $x < 0$? Since $\ln|x| = \ln(-x)$ we have

$$\frac{d}{dx} \ln|x| = \frac{d}{dx} \ln(-x) = \frac{1}{-x}(-1) = \frac{1}{x}$$

So we see that $\ln|x|$ is an antiderivative of $\frac{1}{x}$ when $x < 0$

Combining these results:

$$\int \frac{1}{x} dx = \ln|x| + c, \quad x \neq 0$$

Ex. $\int 5x^{-1} dx = \int \frac{5}{x} dx = 5\ln|x| + c$

Ex Since $\frac{d}{dx} \ln|f(x)| = \frac{1}{f(x)} f'(x) = f'(x)/f(x)$

we have $\int f'(x)/f(x) \, dx = \ln|f(x)| + c$

Ex $\int -\tan x \, dx = \int \frac{-\sin x}{\cos x} dx = \ln|\cos x| + c$

$\int \tan x \, dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| + c$

Ex Suppose a bacteria culture starts with 100 bacteria and 3 hours later there are 230 bacteria. How much time has elapsed when the culture has 1000 bacteria?

Solving this requires a mathematical model. In the case of bacteria, we know that in the presence of a food source and the absence of any limiting factors the rate the culture grows is proportional to the number present.

If $q(t)$ gives the population at time t we can write

$$q(0) = 100$$
$$q(3) = 230$$

Using the proportion described above, we have

$$\frac{dq}{dt} = kq$$

Rearranging, we have

$$\frac{1}{q}\frac{dq}{dt} = k$$

Since $\frac{d}{dt} \ln|q(t)| = \frac{1}{q(t)} \cdot \frac{dq}{dt}$ we can write

$$\frac{d}{dt}\left[\ln q\right] = k$$

What function is needed so that $\frac{d}{dt}\left[\quad\right] = k$?

kt works

so does $kt + c$ for any constant c

$$\therefore \quad \frac{d}{dt}\left[\ln q\right] = \frac{d}{dt}\left[kt + c\right]$$

so $\quad \ln q = kt + c$

Solving for q we have

$$e^{\ln q} = e^{kt+c}$$
$$q(t) = e^{kt}e^{c}$$

Notice that e^c is a constant. This means that $q(t)$ is a constant multiple of e^{kt}, i.e.

$$q(t) = q_0 e^{kt}$$

We use the constant q_0 since this is the value of q when $t=0$

$$q(0) = 100 = q_0 e^{0k} = q_0$$

$$\therefore \quad q(t) = 100 e^{kt}$$

When $t=3$ we have $q(3) = 230 = 100 e^{3k}$

$$2.3 = e^{3k}$$
$$\ln 2.3 = 3k$$

$$k = \frac{\ln 2.3}{3} \approx 0.27764$$

$$\therefore \quad \boxed{q(t) = 100 e^{\frac{t}{3}\ln 2.3}}$$

The time when $q=1000$ can now be found.

$$1000 = 100 e^{\frac{t}{3}\ln 2.3}$$
$$\ln 10 = \frac{t}{3}\ln 2.3$$

$$t = \frac{3\ln 10}{\ln 2.3} = 24.88$$

It will take 24.88 hours (24 hours and almost 53 minutes) for the bacteria population to reach 1000.