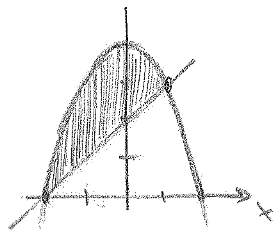
Applications of Integrations

Area and Volume

Ex Find the area between the curves $y = x+2$ and $y = 4-x^2$.



$$A = \int_{-2}^{1} (4-x^2)\,dx - \int_{-2}^{1} (x+2)\,dx$$

$$= \left(4x - \frac{x^3}{3}\right)\Big|_{-2}^{1} - \left[\left(\frac{x^2}{2} + 2x\right)\Big|_{-2}^{1}\right]$$

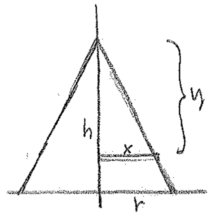
$$= 4 - \frac{1}{3} - \left(4(-2) - \frac{(-2)^3}{3}\right) - \left[\frac{1}{2} + 2 - \left(\frac{(-2)^2}{2} + 2(-2)\right)\right]$$

$$= \cancel{4} - \frac{1}{3} + 8 - \frac{8}{3} - \frac{1}{2} \cancel{-2} \cancel{+2} \cancel{-4} = 8 - 3 - \tfrac{1}{2}$$

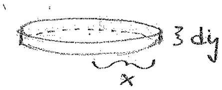
$$= 4.5$$

$\boxed{\text{Area is } 4.5 \text{ sq. units}}$

Ex Compute the volume of a right circular cone of height h and base radius r

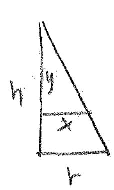


We will compute the volume by "slicing"



$dV = \pi x^2 dy$

Volume of a slice with radius x and thickness dy



Notice that we can relate x and y by similar triangles

$\frac{x}{y} = \frac{r}{h}$ so $x = \frac{r}{h} y$ $\rightarrow$ $dV = \pi \left(\frac{r}{h} y\right)^2 dy$

So, the volume of the cone is the sum of the volumes of the slices:

$$V = \int_0^h \pi \frac{r^2}{h^2} y^2\,dy = \frac{\pi r^2}{h^2} \int_0^h y^2\,dy = \frac{\pi r^2}{h^2} \frac{y^3}{3}\Big|_0^h$$

$$= \frac{\pi r^2}{h^2}\left[\frac{h^3}{3} - 0\right] = \frac{1}{3}\pi r^2 h$$

$$\boxed{V = \frac{1}{3}\pi r^2 h}$$

Applications of Integration

The Future Value after T years of an income stream of P(t) dollars per year assuming continuous compounding at rate r per year is

$$FV = \int_0^T P(t)\, e^{r(T-t)}\, dt$$

Ex A business is expected to generate \$40,000 of income per year for the next 5 years. If this income is reinvested, earning interest of 8% per year compounded continuously, what is the total future value?

$$FV = \int_0^5 40000\, e^{0.08(5-t)}\, dt$$

$$= 40000\, e^{(0.08)(5)} \int_0^5 e^{-0.08t}\, dt$$

$$= 40000\, e^{0.4}\, \frac{e^{-0.08t}}{-0.08} \Big|_0^5$$

$$= -500000\, e^{0.4} \left[e^{-0.4} - e^{0} \right]$$

$$= 500000\, e^{0.4} \left[1 - e^{-0.4} \right] = 500000 \left(e^{0.4} - 1 \right)$$

$$\approx \underline{\$245{,}912.35}$$

The Present Value after T years of an income stream of P(t) dollars per year assuming continuous compounding at rate r per year is

$$PV = \int_0^T P(t)\, e^{-rt}\, dt$$

Ex Compute the present value of the income stream in the last example.

$$PV = \int_0^5 40000\, e^{-0.08t}\, dt$$

$$= 40000 \int_0^5 e^{-0.08t}\, dt$$

$$= 40000\, \frac{e^{-0.08t}}{-0.08} \Big|_0^5$$

$$= -500000 \left(e^{-0.4} - 1 \right) = 500000 \left(1 - e^{-0.4} \right)$$

$$= \underline{\$164{,}839.98}$$