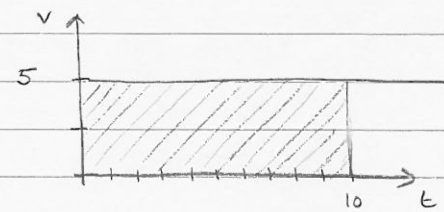
[DEMONSTRATE RIEMANN SUM APPLET!]
(Needed for homework)

Area

Suppose a vehicle is traveling with a velocity $v(t) = 5$ m/sec. How far will it go in 10 seconds?

$$d = r \times t \rightarrow \quad d = 5 \text{ m/sec} \times 10 \text{ sec} = 50 \text{ m}$$

We can draw a picture of this to see that

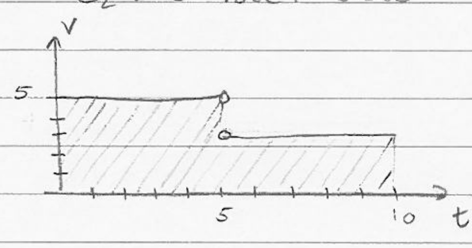


Computing the shaded area gives us the distance travelled.

Now suppose the velocity is given by

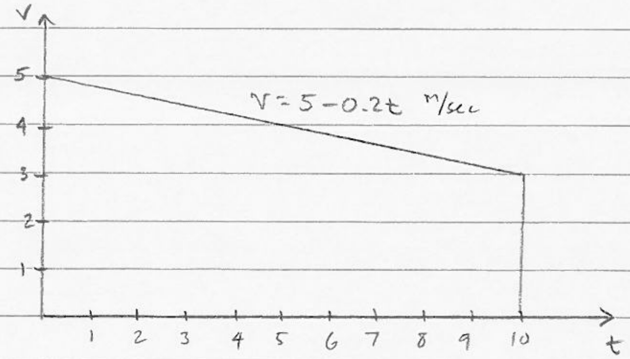
$$v(t) = \begin{cases} 5 \text{ m/sec} & 0 \le t < 5 \\ 3 \text{ m/sec} & 5 \le t \le 10 \end{cases}$$

$$d_1 = 5 \text{ m/sec} \times 5 \text{ sec} = 25 \text{ m}$$
$$d_2 = 3 \text{ m/sec} \times 5 \text{ sec} = 15 \text{ m} \quad \rightarrow \quad d = 40 \text{ m}$$



Once again the area is equivalent to the distance traveled.

Finally, suppose we have $v(t) = 5 - 0.2t$ m/sec.

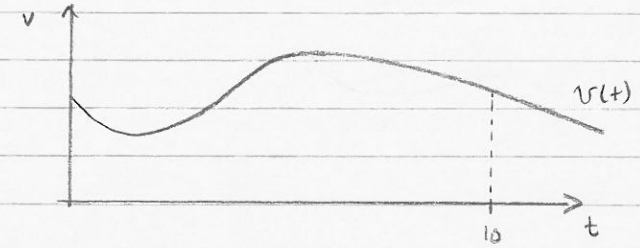


By the same reasoning, the area should give us the distance travelled.

$$A = \frac{5+3}{2} \cdot 10 = 40$$

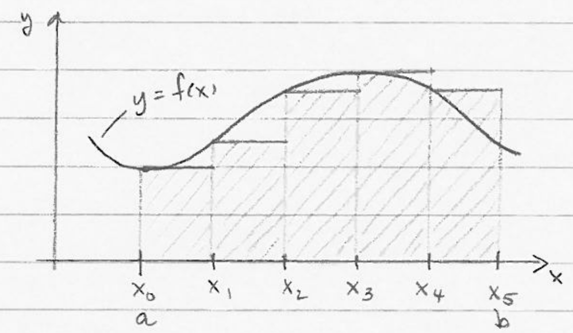
Distance = 40 m

Now, what if $v(t) = f(t)$



We need to find the area between the curve and the t axis between $t=0$ and $t=10$.

We can approximate this area by summations. In general



We divide the interval $[a,b]$ up into n parts [this is called a partition] and approximate the area on each part with a rectangle.

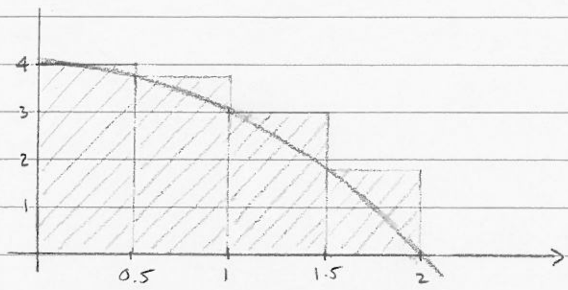
If $\Delta x = x_{i+1} - x_i$ (the width of each part) then

$$A \simeq f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + f(x_4)\Delta x$$

$$= \sum_{i=0}^{4} f(x_i)\Delta x$$

Notice that $\Delta x = \frac{b-a}{n}$ or $n = \frac{b-a}{\Delta x}$

Ex. Approximate the area under the curve $y = 4 - x^2$ on the interval $[0,2]$ using $n = 4$



$f(x) = 4 - x^2$

$\Delta x = \frac{2-0}{4} = \frac{1}{2}$

$x_i = 0 + i\Delta x = i/2$

$$A \approx A_4 = \sum_{i=0}^{3} f(x_i)\Delta x = \sum_{i=0}^{3} \left(4 - \left(\tfrac{i}{2}\right)^2\right)\left(\tfrac{1}{2}\right)$$

$$= (4-0)\left(\tfrac{1}{2}\right) + \left(4-\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + (4-1)\left(\tfrac{1}{2}\right) + \left(4-\tfrac{9}{4}\right)\left(\tfrac{1}{2}\right)$$

$$= 2 + \frac{15}{8} + \frac{3}{2} + \frac{7}{8} = \frac{16+15+12+7}{8} = \frac{50}{8} = \underline{\underline{6.25}}$$

Is this answer too big or too small?

How could we improve it?

$$A \approx A_4 = \sum_{i=1}^{4} f(x_i)\Delta x = \sum_{i=1}^{4} \left(4 - \left(\tfrac{i}{2}\right)^2\right)\left(\tfrac{1}{2}\right)$$

$$= \left(4-\tfrac{1}{4}\right)\left(\tfrac{1}{2}\right) + (4-1)\left(\tfrac{1}{2}\right) + \left(4-\tfrac{9}{4}\right)\left(\tfrac{1}{2}\right) + (4-4)\left(\tfrac{1}{2}\right)$$

$$= \frac{15}{8} + \frac{3}{2} + \frac{7}{8} + 0 = \frac{15+12+7}{8} = \frac{34}{8} = \underline{\underline{4.25}}$$

Is this too big or too small?

Notice the change in the sum $\sum_{i=0}^{n-1}$ to $\sum_{i=1}^{n}$ and the corresponding change in how the heights of the rectangles is computed.

Clearly, as the number of intervals n increases, we have better approximations to the area.

Def. For a function f defined on the interval $[a,b]$, if f is continuous on $[a,b]$ and $f(x) \geq 0$ on $[a,b]$, the <u>area A under the curve</u> $y = f(x)$ on $[a,b]$ is given by

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

Ex. Compute the exact area under the curve $y = 4 - x^2$ on $[0,2]$.

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\,\Delta x$$

where $\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$

left endpoint

and $x_i = 0 + i\Delta x = i\frac{2}{n} = 2i/n$

$\therefore$

$$\begin{aligned}
A &= \lim_{n\to\infty} \sum_{i=1}^{n} \left[4 - \left(\tfrac{2i}{n}\right)^2\right]\left(\tfrac{2}{n}\right) \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{8}{n} - \frac{8i^2}{n^3}\right) \\
&= \lim_{n\to\infty} \frac{8}{n}\cdot n - \frac{8}{n^3}\sum_{i=1}^{n} i^2 \\
&= 8 - \lim_{n\to\infty} \frac{8}{n^3}\,\frac{n(n+1)(2n+1)}{6} \\
&= 8 - \lim_{n\to\infty} \frac{8}{n^2}\,\frac{2n^2+3n+1}{6} \\
&= 8 - \lim_{n\to\infty} \frac{8}{6}\left(2 + \frac{3}{n} + \frac{1}{n^2}\right) \\
&= 8 - \frac{16}{6} \\
&= 8 - \frac{8}{3} \\
&= 16/3 = 5.\overline{3}
\end{aligned}$$

Def. Let $\{x_0, x_1, \ldots, x_n\}$ be a regular partition of the interval $[a,b]$ with $x_i - x_{i-1} = \Delta x = \frac{b-a}{n}$, for all i.

Pick points $c_1, c_2, \ldots, c_n$ where $c_i \in [x_{i-1}, x_i]$, for $i = 1, 2, \ldots, n$.

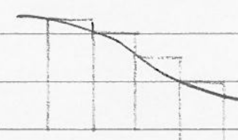
The <u>Riemann Sum</u> for this partition and set of evaluation points c_i is

$$\sum_{i=1}^{n} f(c_i)\,\Delta x.$$

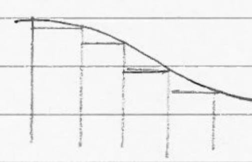
A "regular partition" just means all subintervals are the same size.

We are free to choose c_i as we see fit. The most common choices are

$c_i = x_{i-1}$ (left-hand sum)



$c_i = x_i$ (Right hand sum)



$c_i = \frac{x_{i-1} + x_i}{2}$ (midpoint)