The Chain Rule

Suppose we have a mixture containing 3 g/ℓ of salt and that this mixture is flowing into a tank at a rate of 5 ℓ/min.
How quickly is salt entering the tank?

$$(3\ g/\ell)(5\ \ell/min) = 15\ g/min$$

Salt is entering the tank at a rate of 15 g/min.

Now suppose dm/dv is the rate at which salt is disolved in a volume of water and dv/dt is the rate the mixture is flowing into a tank. Then the rate at which salt is flowing into the tank is

$$dm/dv \cdot dv/dt = dm/dt \quad (g/min)$$

Chain Rule - Version One

$$\boxed{\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx}}$$

Suppose $h(x) = \sqrt{x^2+1}$ and we want to find $h'(x)$.

Using the definition is possible but is fairly complicated. We can, however, use the chain rule.

Let $f(z) = \sqrt{z}$ and $g(x) = x^2+1$ so $h(x) = f(g(x))$.
Recall that this means $h(x)$ is a composite function.

Let $y = h(x)$. Then $y = f(z)$ if $z = g(x)$ since then $y = f(g(x))$.
So

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \left(\frac{d}{dz} f(z)\right)\left(\frac{d}{dx} g(x)\right)$$

$$= \left(\frac{d}{dz} \sqrt{z}\right)\left(\frac{d}{dx}(x^2+1)\right)$$

$$= \frac{1}{2\sqrt{z}} \cdot (2x)$$

We finish up by replacing $z = x^2+1$ so we have

$$h'(x) = \frac{2x}{2\sqrt{x^2+1}} = \frac{x}{\sqrt{x^2+1}}.$$

Well, it wasn't pretty, but we did get the correct derivative.

Notice what we did...

$$\frac{dy}{dx} = \frac{d}{dx} f(g(x)) = \frac{d}{dz} f(z) \cdot \frac{d}{dx} g(x) \qquad \text{since } z = g(x)$$

$$= f'(z)\, g'(x)$$

$$= f'(g(x))\, g'(x)$$

Chain Rule - Version Two

$$\boxed{\frac{d}{dx} f(g(x)) = f'(g(x)) \cdot g'(x)}$$

Ex $\frac{d}{dx}(x^3+2x)^2$ Here $f(z) = z^2$, $z = g(x) = x^3+2x$

$f(z)$ is the "outside" function, $g(x)$ is the "inside" function.

$$\frac{d}{dx}(x^3+2x)^2 = \underbrace{2(x^3+2x)^1}_{f'(g(x))} \cdot \underbrace{(3x^2+2)}_{g'(x)}$$

Ex Let $f(x) = \dfrac{x^3}{(1-x^2)^2}$

$$f'(x) = \frac{(1-x^2)^2 \cdot (3x) - \overbrace{2(1-x^2)^1 \cdot (-2x)}^{\text{chain rule used here.}} \cdot x^3}{(1-x^2)^4}$$

The Chain Rule

Suppose we have a function $f(x)$ that is differentiable and invertible at x.

Let $g(x) = f^{-1}(x)$ so $f(g(x)) = f(f^{-1}(x)) = x$

Then

$$\frac{d}{dx} f(g(x)) = \frac{dx}{dx} = 1$$

$$f'(g(x))\, g'(x) = 1$$

so $$g'(x) = 1 / f'(g(x))$$

$$\boxed{\frac{d}{dx} f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}}$$

Ex. $f(x) = x^2$, $x > 0$. Then $f^{-1}(x) = g(x) = \sqrt{x}$

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2(\sqrt{x})'} = \frac{1}{2\sqrt{x}}$$

which we already know is the proper derivative of $\sqrt{x}$.