Computation of Limits

Consider $\lim_{x\to 2} (x^2-4)/(x-2)$. We saw this limit in the last lecture.

Using algebra, we were able to simplify $(x^2-4)/(x-2)$

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

So

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} x+2$$

and as $x \to 2$ we find that $x+2 \to 4$.

In this lecture we will see how to evaluate limits like this, using algebra and arithmetic rather than a graph or numerical work.

We begin with a list of simple limits and theorems that will help us.

1. For any constant c and any real number a

$$\lim_{x\to a} c = c$$

2. For any real number a

$$\lim_{x\to a} x = a$$

The next four properties are presented as Theorem 3.1 in our text.

Suppose that $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and let c be any constant. Then

3. $\lim_{x\to a} [c\, f(x)] = c \cdot \lim_{x\to a} f(x)$

4. $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$

Computation of Limits

5. $\lim_{x\to a} [f(x)\cdot g(x)] = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)]$

6. $\lim_{x\to a} f(x)/g(x) = [\lim_{x\to a} f(x)]/[\lim_{x\to a} g(x)]$ provided $\lim_{x\to a} g(x) \neq 0$.

Ex $\lim_{x\to 1} 3x+2 = \lim_{x\to 1} 3x + \lim_{x\to 1} 2$ by #4

$= \lim_{x\to 1} 3x + 2$ by #1

$= [\lim_{x\to 1} 3][\lim_{x\to 1} x] + 2$ by #5

$= 3\cdot 1 + 2$ by #1, 2

$= \boxed{5.}$

Ex $\lim_{x\to 5} \frac{2x^2+1}{x-3} = (\lim_{x\to 5} 2x^2+1)/(\lim_{x\to 5} x-3)$ by #6

$= \dfrac{\lim_{x\to 5} 2x + \lim_{x\to 5} 1}{\lim_{x\to 5} x - \lim_{x\to 5} 3}$ by #4

$= \dfrac{2\lim_{x\to 5} x + 1}{5-3}$ by #3, 1 by #2, 1

$= \dfrac{2\cdot 5+1}{2}$ by #2

$= \boxed{\dfrac{11}{2}}$

Ex $\lim_{x\to 1} (x^3-1)/(x-1) = \lim_{x\to 1} \frac{(x-1)(x^2+x+1)}{(x-1)}$

$= \lim_{x\to 1} x^2+x+1$

$= \lim_{x\to 1} x^2 + \lim_{x\to 1} x + \lim_{x\to 1} 1$ by #4

$= (\lim_{x\to 1} x)(\lim_{x\to 1} x) + 1 + 1$ by #5, 2, 1

$= 1\cdot 1+1+1 = \boxed{3}$

Computation of Limits

By now you may have noticed that the limits of polynomials are easy.

Thm: For any polynomial $p(x)$ and any real number a

$$\lim_{x\to a} p(x) = p(a)$$

Proof.

Let $p(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$

Then

$$\lim_{x\to a} c_n x^n + c_{n-1}x^{n-1} + \cdots + c_1 x + c_0$$

$$= \lim_{x\to a} c_n x^n + \lim_{x\to a} c_{n-1}x^{n-1} + \cdots + \lim_{x\to a} c_1 x + \lim_{x\to a} c_0 \qquad \text{by \#4}$$

$$= c_n \lim_{x\to a} x^n + c_{n-1}\lim_{x\to a} x^{n-1} + \cdots + c_1 \lim_{x\to a} x + c_0 \qquad \text{by \#3, 1}$$

$$= c_n (\lim_{x\to a} x)^n + c_{n-1}(\lim_{x\to a} x)^{n-1} + \cdots + c_1 \lim_{x\to} x + c_0 \qquad \text{by \#5 (repeated)}$$

$$= c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 \qquad \text{by \#2}$$

$$= p(a) \quad \checkmark$$

Ex $\lim_{x\to -2} x^2 - x + 7 = (-2)^2 - (-2) + 7$

$= 4 + 2 + 7$

$= \boxed{13}$

Thm: Suppose that $\lim_{x\to a} f(x) = L$ and n is any positive integer. Then

$$\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L}$$

where for n even we assume $L > 0$.

Ex $\lim_{x\to 3} \sqrt{x^3 - 2x} = \sqrt{\lim_{x\to 3} x^3 - 2x} = \sqrt{27-6} = \boxed{\sqrt{21}}$

Ex $\lim_{x\to 0} \frac{x}{\sqrt{x+9}-3}$ We can rationalize the denominator

$$= \lim_{x\to 0} \frac{x}{\sqrt{x+9}-3} \cdot \frac{\sqrt{x+9}+3}{\sqrt{x+9}+3}$$

$$= \lim_{x\to 0} \frac{x(\sqrt{x+9}+3)}{x+9-9}$$

$$= \lim_{x\to 0} \frac{x(\sqrt{x+9}+3)}{x}$$

$$= \lim_{x\to 0} \sqrt{x+9}+3$$

$$= \sqrt{9}+3 = \boxed{6}$$

All the functions $\sin x$, $\cos x$, $\tan x$, e^x, $\ln x$, $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ are continuous, a property we'll look at in depth soon. One consequence of this is that if $f(x)$ is any of these function and a is any real number in the domain of f then

$$\lim_{x\to a} f(x) = f(a).$$

Ex $\lim_{x\to 2^-} \ln \frac{2-x}{2-\sqrt{x}}$ Note: this is a one sided limit because natural log is only defined for positive arguments.

$$\lim_{x\to 2^-} \ln \frac{2-x}{2-\sqrt{x}} = \lim_{x\to 2^-} \frac{2-x}{(2-\sqrt{x})} \frac{(2+\sqrt{x})}{(2+\sqrt{x})}$$ rationalization

$$= \lim_{x\to 2^-} \frac{(2-x)(2+\sqrt{x})}{4-x}$$ "

$$= \frac{\lim_{x\to 2^-} (2-x)(2+\sqrt{x})}{\lim_{x\to 2^-} 4-x}$$ by #6

$$= \frac{0}{4} = \boxed{0}$$ by polynomial thm.

Computation of Limits

Ex Evaluate $\lim_{x\to 0} x/\sqrt{x}$ and $\lim_{x\to 0} \sqrt{x}/x$

$$\lim_{x\to 0} x/\sqrt{x} = \lim_{x\to 0} \sqrt{x} = 0$$

$$\lim_{x\to 0} \sqrt{x}/x = \lim_{x\to 0} \frac{1}{\sqrt{x}} \quad \text{does not exist}$$

Notice the first limit $\lim_{x\to 0} x/\sqrt{x}$ can be written $\lim_{x\to 0}\left(x\cdot\frac{1}{\sqrt{x}}\right)$ so

$$\lim_{x\to 0}\left(x\cdot\frac{1}{\sqrt{x}}\right) = 0$$

This does not mean, however, that both $\lim_{x\to 0} x$ and $\lim_{x\to 0}\frac{1}{\sqrt{x}}$ exist.

Thm (Squeeze or Sandwich Thm)

Suppose that $f(x) \le g(x) \le h(x)$ for all x in some interval (c,d), except possibly at the point $a\in(c,d)$ and that

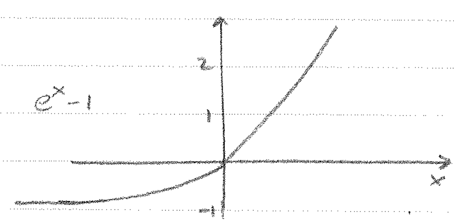
$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

for some number L. Then

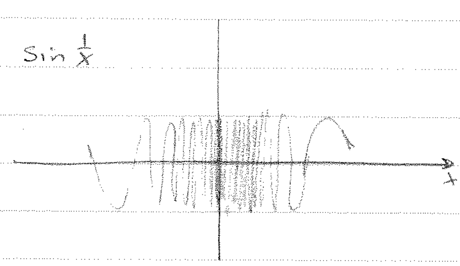
$$\lim_{x\to a} g(x) = L.$$

This very useful theorem allows us to find the limit of many functions that we cannot deal with in other ways.

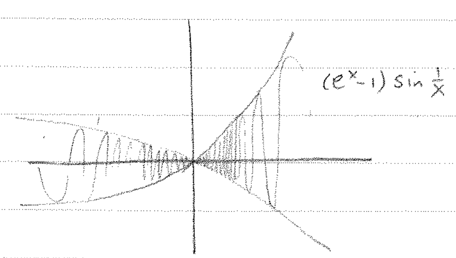
Ex $\lim_{x\to 0^+} (e^x-1)\sin\frac{1}{x}$



$$\lim_{x\to 0^+}(e^x-1) = 0$$



$\lim_{x\to 0^+}\sin\frac{1}{x}$ d.n.e



Notice that on the interval $(0, \infty)$ we have

$$-(e^x - 1) \leq (e^x - 1)\sin\frac{1}{x} \leq (e^x - 1)$$

Since $\lim_{x \to 0^+} \left[-(e^x - 1)\right] = 0$ and $\lim_{x \to 0^+} e^x - 1 = 0$

by the squeeze theorem we know $\lim_{x \to 0^+} (e^x - 1)\sin\frac{1}{x} = 0$.