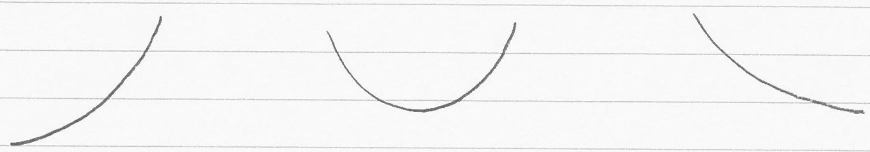
Concavity and the Second Derivative Test

The first derivative of f measures the rate of change of f – i.e. it tells how fast f is changing as the input to f changes.

What does f'' measure? The rate of change of f'.

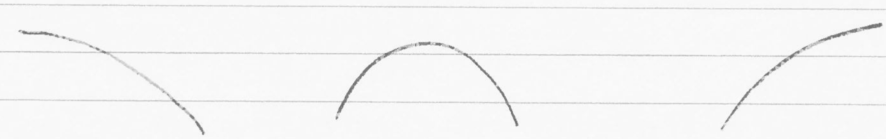
If $f''(x) > 0$ then $f'(x)$ is increasing. Possible curves for this f are



Notice that these may be increasing or decreasing ($f'(x) > 0$ or $f'(x) < 0$) but in each case $f''(x) > 0$ and $f'(x)$ is increasing

We say these curves are <u>concave up.</u>

If $f''(x) < 0$ then $f'(x)$ is decreasing and could look like

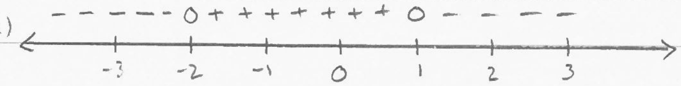


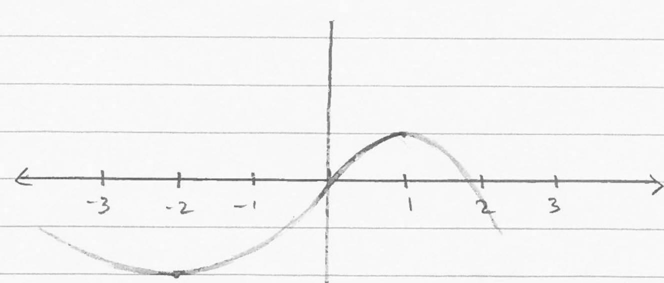
These curves are <u>concave down.</u>

Ex. Sketch the graph of a function with the signs of $f'(x)$ and $f''(x)$ given by

$f''(x)$ + + + + + + + + 0 − − − − − − − −

$f'(x)$ − − − − − 0 + + + + + + + 0 − − − − −





Concavity and the Second Derivative Test

A point where the concavity changes is called a <u>point of inflection</u>, or an <u>inflection point</u>.

Ex Find the points of inflection of $f(x) = 2x^3 - 3x^2 + 5x - 2$

$$f'(x) = 6x^2 - 6x + 5$$
$$f''(x) = 12x - 6$$

For the concavity to change, the second derivative must change sign. Since this $f''(x)$ is continuous, the concavity can only change where $f''(x) = 0$.

$$12x - 6 = 0 \rightarrow x = \tfrac{1}{2}$$
$$f(\tfrac{1}{2}) = 2 \cdot \tfrac{1}{8} - 3 \cdot \tfrac{1}{4} + 5 \cdot \tfrac{1}{2} - 2$$
$$= \frac{1}{4} - \frac{3}{4} + \frac{10}{4} - \frac{8}{4}$$
$$= 0$$

Note that when $x < \frac{1}{2}$, $f''(x) < 0$ and when $x > \frac{1}{2}$ $f''(x) > 0$.

The point of inflection is $(\frac{1}{2}, 0)$.

Ex. Find the points of inflection of $f(x) = \sin x$

$$f'(x) = \cos x, \qquad f''(x) = -\sin x$$

Since $\sin x = 0$ if $x = n\pi$ for all integers n and we know from our graphs of $\sin x$ that the sign of sine changes at these points, we find the the points of inflection are

$$(n\pi, 0) \qquad n = 0, \pm 1, \pm 2, \ldots$$

Thm Second Derivative Test.

Suppose that f is continuous on the interval (a,b) and $f'(c) = 0$ for some number $c \in (a,b)$.

1) If $f''(c) < 0$ then $f(c)$ is a local maximum
2) If $f''(c) > 0$ then $f(c)$ is a local minimum
3) If $f''(c) = 0$ the test is inconclusive

Ex Use the second derivative test to find the local extrema of $f(x) = 2x^4 - 8x^2 + 1$

$f'(x) = 8x^3 - 16x$ $\qquad$ $f''(x) = 24x^2 - 16$

$f'(x) = 0 \Rightarrow 8x^3 - 16x = 0$

$8x(x^2 - 2) = 0$ $\qquad$ $x = 0,\ x = -\sqrt{2},\ x = +\sqrt{2}$

$f''(0) = -16 < 0 \Rightarrow$ $\underline{f(0) \text{ is a local maximum}}$

$f''(-\sqrt{2}) = 24 \cdot 2 - 16 = 32 > 0$ $\underline{f(-\sqrt{2}) \text{ is a local minimum}}$

$f''(\sqrt{2}) = 24 \cdot 2 - 16 = 32 > 0$ $\underline{f(\sqrt{2}) \text{ is a local minimum.}}$