The Concept of Limit

What do we know about the function $f(x) = (x^2-4)/(x-2)$?

- Rational
- Domain is $\mathbb{R}$ except $x=2$: $D = \{x \in \mathbb{R} \mid x \neq 2\}$

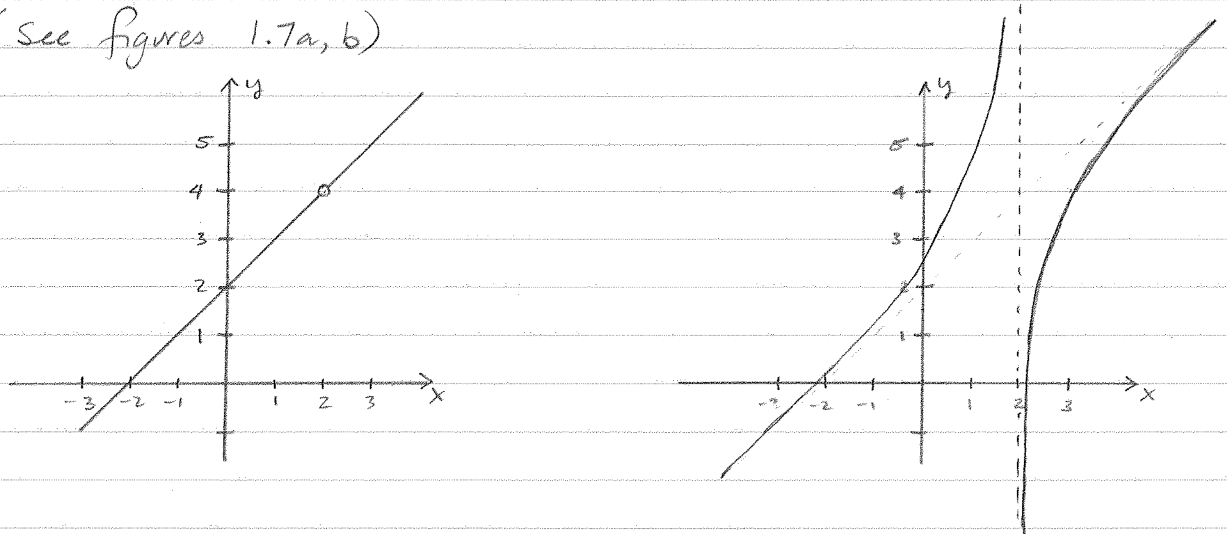
What about the function $g(x) = (x^2-5)/(x-2)$?

- Rational
- $D = \{x \in \mathbb{R} \mid x \neq 2\}$

Does this suggest that these two functions are similar?

Start by examining the functions for x near 2.

(See figures 1.7a, b)



Obviously, the behavior of these two functions near $x=2$ is very different.

It appears that $f(x) = (x^2-4)/(x-2)$ is merely missing a point at $x=2$ while $g(x) = (x^2-5)/(x-2)$ approaches either ∞ or $-\infty$ as x gets close to 2.

It turns out that exploring the behavior of functions as x "gets close to" some particular value is quite important.

Let's focus on $f(x) = (x^2-4)/(x-2)$ for the moment. Notice that

$$\frac{x^2-4}{x-2} = \frac{(x-2)(x+2)}{x-2} = x+2$$

Notice that this <u>does not</u> say $f(x) = x+2$. This is because by writing $(x-2)$ in the denominator we must restrict the domain so $x \neq 2$ — once it's out, it's out!

So, what does happen near $x=2$? If we compute values of f for $x<2$ and $x>2$ we find

x	$f(x)$	x	$f(x)$
1.9	3.9	2.1	4.1
1.99	3.99	2.01	4.01
1.999	3.999	2.001	4.001

It certainly appears to be the case that $f(x)$ gets close to 4 as x gets close to 2.

This brings us to the idea of <u>Limit</u>. We will soon define this very precisely, but for now the intuitive idea we develop here will be sufficient.

We saw that as x approached 2 from the left (or from below), $f(x)$ approached 4. We write this

$$\lim_{x\to 2^-} f(x) = 4$$

which we read as "the limit of $f(x)$ as x goes to 2 from the left (or from below) is 4."

Similarly, we can write $\lim_{x\to 2^+} f(x) = 4$ since we see that as x approaches 2 from the right (or from above) $f(x)$ goes to 4.

The Concept of Limit

Since $\lim\limits_{x \to 2^-} f(x) = \lim\limits_{x \to 2^+} f(x)$ it makes sense to say that the limit of f(x) as x goes to 2 is 4;

$$\lim_{x \to 2} f(x) = 4.$$

We can only do this if both the left-hand limit and the right hand limit are equal.

$$\boxed{\lim_{x \to a} f(x) = L \quad \text{iff} \quad \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L}$$

Next, let's return to $g(x) = (x^2-5)/(x-2)$. What is $\lim\limits_{x \to 2^-} g(x)$? How about $\lim\limits_{x \to 2^+} g(x)$?

x	g(x)
1.9	13.9
1.99	103.99
1.999	1003.999

x	g(x)
2.1	−5.9
2.01	−95.99
2.001	−995.999

It does not appear that g(x) approaches any particular value as $x \to 2^-$ or $x \to 2^+$. As a result, we say

$$\lim_{x \to 2^-} g(x) \quad \text{does not exist (dne)}$$

$$\lim_{x \to 2^+} g(x) \quad \text{does not exist}$$

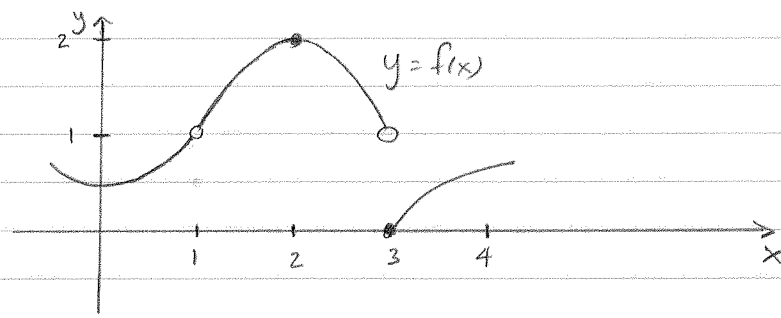
And so

$$\lim_{x \to 2} g(x) \quad \text{does not exist.}$$

We can compute limits several ways. We've just seen how we can compute them <u>numerically</u> (or at least compute what the limit seems to be).

We can also find limits <u>graphically</u>.

<u>Ex</u> Find the left- & right-hand limits of $f(x)$ at $x=1$, $x=2$, and $x=3$.



$\lim_{x\to 1^-} f(x) = 1$, $\lim_{x\to 1^+} f(x) = 1$ so $\lim_{x\to 1} f(x) = 1$

This limit exists even though $f(1)$ is not defined.

$\lim_{x\to 2^-} f(x) = 2$, $\lim_{x\to 2^+} f(x) = 2$. so $\lim_{x\to 2} f(x) = 2$

Notice that $f(2) = 2$ as well.

$\lim_{x\to 3^-} f(x) = 1$, $\lim_{x\to 3^+} f(x) = 0$ so $\lim_{x\to 3} f(x)$ does not exist.

A very important limit is

$$\lim_{x\to 0} \frac{\sin x}{x}$$

We can graph this and investigate it numerically, but it will be some time before we have the necessary tools to prove the value of this limit

Both the graph and numerical experiments suggest that

$\lim_{x \to 0^-} \frac{\sin x}{x} = 1$, $\lim_{x \to 0^+} \frac{\sin x}{x} = 1$ so we say

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

This result leads to the "small angle approximation" in physics,

$\sin x \approx x$ when x is small.

(Note – this is only true if x is in radians and not degrees).

e.g. $\sin(0.01) \simeq 0.0099998333$