Continuity and its Consequences

Ask students if they have an idea of what a continuous function is.

Give them one minute to write a definition.

Give them another minute (or so) to write a defintion in terms of limits.

DEF A function f is continuous at $x=a$ when
i) $f(a)$ is defined,
(ii) $\lim_{x\to a} f(x)$ exists, and
(iii) $\lim_{x\to a} f(x) = f(a)$

Thm: All polynomials are continuous everywhere.

Proof: We've already proven $\lim_{x\to a} p(x) = p(a)$ for every polynomial $p(x)$ and every real number a. This, along with its consequences - that $p(a)$ is defined and that $\lim_{x\to a} p(x)$ exists complete the proof.

Thm: The following functions are all continuous on the indicated intervals.

$\sin x$, $\cos x$, $\tan^{-1} x$, e^x : all $\mathbb{R}$
$\sqrt[n]{x}$: all $\mathbb{R}$ if n odd, all $x>0$ if n even.
$\ln x$: all $x>0$
$\sin^{-1} x$, $\cos^{-1} x$: $-1 \leq x \leq 1$

Thm: Suppose that f and g are continuous at $x=a$. Then all of the following are true
i) $(f \pm g)$ is continuous at $x=a$
(ii) $(f \cdot g)$ is continuous at $x=a$
(iii) (f/g) is continuous at $x=a$ if $g(a) \neq 0$.

Proof: $\lim_{x\to a} [f(x) \pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
$= f(a) \pm g(a)$
$= (f \pm g)(a)$
$\therefore f \pm g$ is continuous

Ex Determine where $\tan x$ is continuous.

- $\tan x = \sin x / \cos x$
- $\sin x$ and $\cos x$ are continuous for all x
- only possible points of discontinuity occur when $\cos x = 0$ which happens when $x = (2n+1)\frac{\pi}{2}$, odd multiples of $\frac{\pi}{2}$
- Checking from the graph of $\tan x$ we see that the limit of $\tan x$ as $x \to (2n+1)\frac{\pi}{2}$ fails to exist

$\therefore$ $\tan x$ is continuous everywhere except at $(2n+1)\frac{\pi}{2}$, $n = 0, \pm 1, \pm 2, \ldots$

Ex. Determine where $(x^2-4)/(x-2)$ is continuous.

Since polynomials are continuous everywhere, the only place where a discontinuity could occur is at $x = 2$.

Since $(x^2-4)/x-2$ does not exist when $x=2$, we conclude the function is not continuous at $x=2$.

$\therefore$ $(x^2-4)/(x-2)$ is continuous everywhere except at $x=2$, i.e. $(-\infty, 2) \cup (2, \infty)$

Ex Notice that we can create a continuous function that matches $(x^2-4)/(x-2)$ everywhere this function exists. We do this by noting

$$\lim_{x \to 2} (x^2-4)/(x-2) = \lim_{x \to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \to 2} x+2 = 4$$

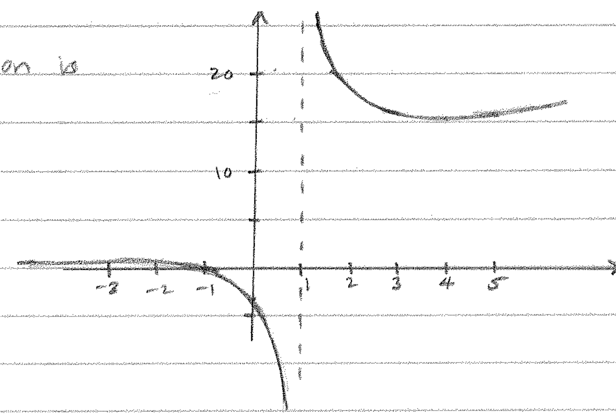
and defining

$$g(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{when } x \neq 2 \\ 4 & \text{when } x = 2 \end{cases}$$

Now, $\lim_{x \to 2} g(x) = g(2)$.

Ex Where is $\dfrac{x^3+2x^2-5x-6}{x^2-3x+2}$ continuous?

According to the graph this function is continuous everywhere except at $x=1$



We should suspect, however, that there may be at least one other point of discontinuity because the denominator is quadratic.

$$x^2-3x+2 = (x-1)(x-2)$$

Sure enough, our function is not continuous at $x=2$ because it does not exist there.

Why doesn't this show up on the graph?

We can try to divide x^3+2x^2-5x-6 by $(x-2)$ to see if $x-2$ is a factor — this would explain why we didn't see it in the graph.

$$\begin{array}{r|l} & x^2+4x+3 \\ \hline x-2 & x^3+2x^2-5x-6 \\ & \underline{x^3-2x^2} \\ & \quad\; 4x^2-5x \\ & \quad\; \underline{4x^2-8x} \\ & \qquad\quad 3x-6 \\ & \qquad\quad \underline{3x-6} \\ & \qquad\qquad 0 \end{array}$$

So $x^3+2x^2-5x-6 = (x-2)(x^2+4x+3)$
$= (x-2)(x+1)(x+3)$

Sure enough, $x-2$ is a factor of the numerator.

The original function is continuous on $(-\infty,1)\cup(1,2)\cup(2,\infty)$.

Continuity and its Consequences

Recall that $(f \circ g)(x)$ means $f(g(x))$.

Thm: Suppose that $\lim_{x\to a} g(x) = L$ and that f is continuous at L. Then

$$\lim_{x\to a} (f\circ g)(x) = \lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right) = f(L)$$

Corollary: Suppose that g is continuous at a and that f is continuous at $g(a)$. Then $f\circ g$ is continuous at a.

<u>Ex</u> $\lim_{x\to 2} e^{x^2-4} = e^{\lim_{x\to 2} x^2-4} = e^0 = \boxed{1}$

So far we have only talked about continuity at a point. We now extend the idea to an interval.

Def: If f is continuous at every point on an open interval (a,b), we say that <u>f is continuous on (a,b)</u>.

Def We say <u>f is continuous on the closed interval $[a,b]$</u> if f is continuous on (a,b) and

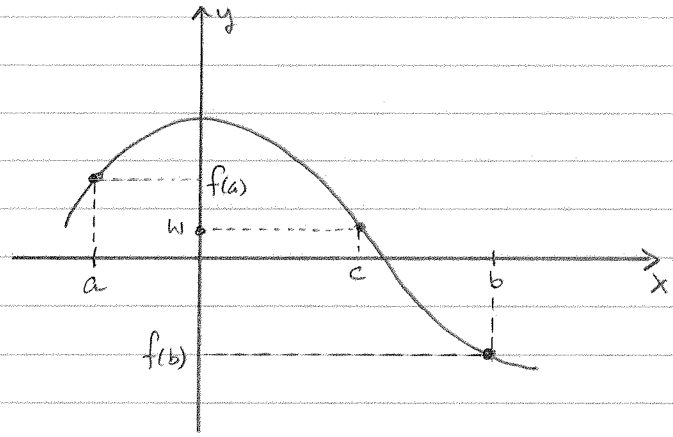
$$\lim_{x\to a^+} f(x) = f(a) \quad \text{and} \quad \lim_{x\to b^-} f(x) = f(b)$$

Def: If f is continuous for all $x\in\mathbb{R}$ then we say <u>f is continuous</u>.

We conclude with an important theorem.

<u>Intermediate Value Theorem</u>

Suppose that f is continuous on the closed interval $[a,b]$ and W is any number between $f(a)$ and $f(b)$. Then there is a number $c\in[a,b]$ for which $f(c)=W$.



As long as $f(a) \le w \le f(b)$ (or $f(b) \le w \le f(a)$) we must find that f takes on the value w at some point in (a,b) because f takes on all possible values between $f(a)$ and $f(b)$.

Ex One important operation in numerical mathematics is finding the zeros of a function.

If f is a continuous function and a and b can be found so

$$f(a)\,f(b) < 0$$

i.e. $f(a)$ & $f(b)$ have opposite sign

then we know that some c exists between a and b for which $f(c)=0$.

Bisection Algorithm:

```
Let a<b with f(a)·f(b) < 0
Repeat
    c = (a+b)/2

    if f(a)f(c) < 0 then b=c   else a=c

Until b-a < maximum-error
c = (a+b)/2
```

Ex $f(x) = x^2 - 2$. Root is $\sqrt{2}$. Start with $a=1$, $b=2$

After 16 iterations, $c = 1.41420746$

$\sqrt{2} = 1.41421356$