Overview of Curve Sketching

Ex Sketch $y = x^3 - 3x^2 - 9x + 9$ showing all significant features

① Since $f(x) = x^3 - 3x^2 - 9x + 9$ is a polynomial we know that the domain of f is all real numbers

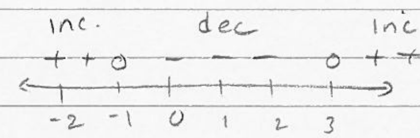
② Find critical numbers

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3)$$
$$= 3(x+1)(x-3)$$

So $x = -1$, $x = 3$ are critical numbers

③ Determine sign of $f'(x)$

$f'(-2) > 0$
$f'(0) < 0$
$f'(4) > 0$

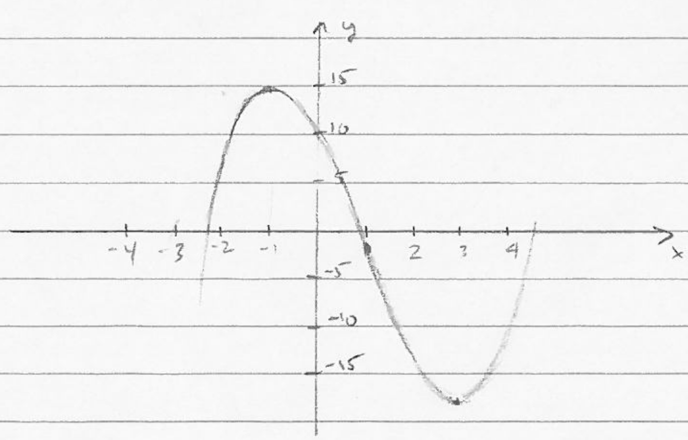


④ Identify local extrema and inflection points

$$f''(x) = 6x - 6 = 6(x-1)$$

<u>Inflection point</u> at $x = 1$ since $f''(x) < 0$ if $x < 1$
$f''(x) > 0$ if $x > 1$

$f(-1)$ is a local maximum $(-1, 14)$
$f(3)$ is a local minimum $(3, -18)$
$f(1) = -2$ so $(1, -2)$ is an inflection point



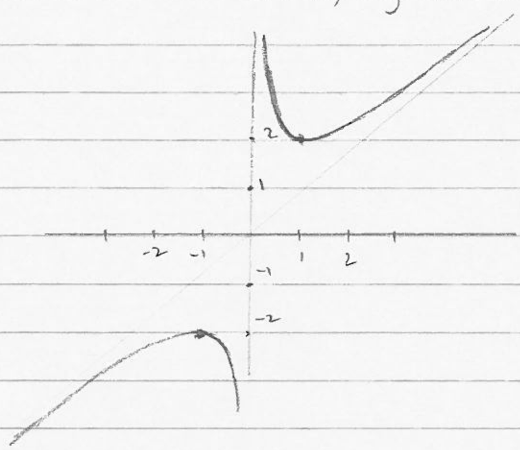
Overview of Curve Sketching

Ex $y = \frac{x^2+1}{x}$ defined for all $x \neq 0$ → Vertical asymptote at $x=0$

$y' = \frac{x(2x) - (x^2+1)}{x^2} = \frac{x^2-1}{x^2} = 1 - \frac{1}{x^2}$ critical numbers ± 1.

$y'' = 2x^{-3} = 2/x^3$ no points of inflection

$x=-1,\ y=-2$ $y''<0$ local max
$x=1,\ y=2$ $y''>0$ local min



We suspect an oblique asymptote so divide to get

$y = x + 1/x$

Local max < local min

Ex $y = x^{5/3} + 5x^{2/3}$
$= x^{2/3}(x+5)$

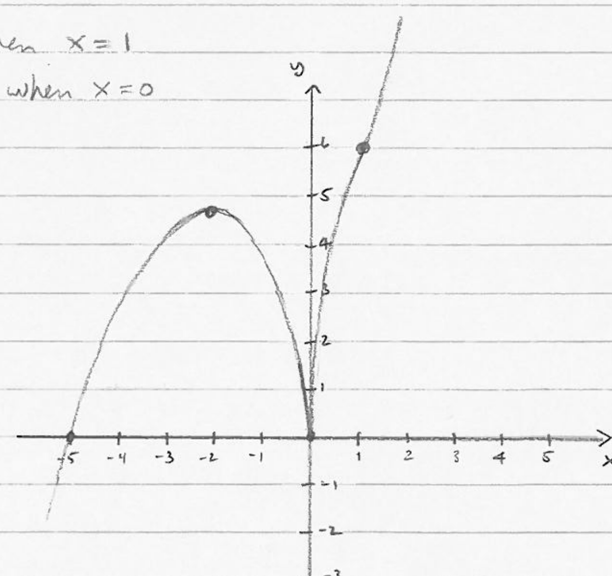
$y' = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3}$

$y'' = \frac{10}{9}x^{-1/3} - \frac{10}{9}x^{-4/3}$

$y' = \frac{5x+10}{3x^{1/3}}$ zero when $5x+10=0$ → $x=-2$
undefined when $x=0$

$y'' = \frac{10}{9}\left(\frac{x-1}{x^{4/3}}\right)$ zero when $x=1$
undefined when $x=0$

	$f(x)$	$f'(x)$	$f''(x)$	Shape
$x\in(-\infty,-2)$		+	−	inc. down
$x=-2$	$3(2)^{2/3}$	0	−	loc max
$x\in(-2,0)$		−	−	dec down
$x=0$	0	und.	und.	
$x\in(0,1)$		+	−	inc down
$x=1$	6	+	0	inf. pt
$x\in(1,\infty)$		+	+	inc. up



Ex. $f(x) = \dfrac{x - \sqrt{x^2+1}}{x} = 1 - \dfrac{\sqrt{x^2+1}}{x}$ $\qquad x \neq 0$

$$f'(x) = -\frac{x\frac{1}{2}(x^2+1)^{-1/2}(2x) - (x^2+1)^{1/2}}{x^2} = -\frac{x^2-(x^2+1)}{x^2\sqrt{x^2+1}} = \frac{1}{x^2\sqrt{x^2+1}}$$

$$f''(x) = \frac{0 - \left[2x\sqrt{x^2+1} + \frac{1}{2}x^2(x^2+1)^{-1/2}(2x)\right]}{x^4(x^2+1)} = \frac{-2x(x^2+1) - x^3}{x^4(x^2+1)^{3/2}} = -\frac{3x^2+2}{x^3(x^2+1)^{3/2}}$$

		0	
$f(x)$			
$f'(x)$	+ + +		+ + +
$f''(x)$	+ + +		− − −

f, f', f'' all undefined at $x = 0$

As $x \to +\infty$ we know $x > 0$ so $f(x) = 1 - \sqrt{\dfrac{x^2+1}{x^2}} = 1 - \sqrt{1+\frac{1}{x^2}}$
so $f(x) \to 1 - 1 = 0$.

As $x \to -\infty$ we have $f(x) = 1 + \dfrac{\sqrt{x^2+1}}{|x|} = 1 + \sqrt{\dfrac{x^2+1}{x^2}} = 1 + \sqrt{1+\frac{1}{x^2}}$
so $f(x) \to 1 + 1 = 2$.

As $x \to 0^+$ $\quad 1 - \sqrt{\dfrac{x^2+1}{x^2}} = 1 - \sqrt{1+\frac{1}{x^2}} \to 1 - \infty = -\infty$

As $x \to 0^-$ $\quad 1 - \dfrac{\sqrt{x^2+1}}{x} = 1 + \dfrac{\sqrt{x^2+1}}{|x|} = 1 + \sqrt{1+1/x^2} = 1 + \infty = \infty$