The Definite Integral

We're seen that to define the area under the curve $y = f(x)$ on the interval $[a,b]$ we can use the limit as $n \to \infty$ of the sum of the area of n rectangles defined on a partition of the interval.

$$A = \lim_{n \to \infty} \sum_{n=1}^{n} f(x_i)\,\Delta x$$

Notice that the sum is finite, but the area is the value the sums approach as we recompute them with larger and larger values of n.

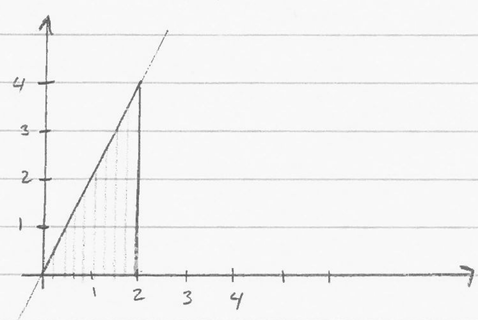
This operation is so important to our work that we define special notation for it

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \sum_{n=1}^{n} f(c_i)\,\Delta x$$

Where $[a,b]$ has been partitioned into n subintervals $[x_{i-1}, x_i]$ each of width Δx and height $f(c_i)$, $c_i \in [x_{i-1}, x_i]$

Ex $\displaystyle\int_0^2 2x\,dx$

Here we can compute the area directly

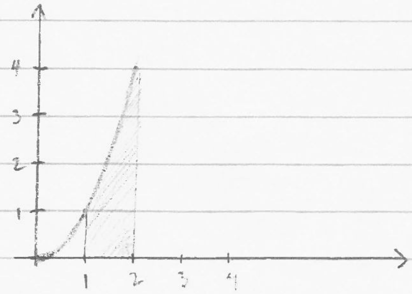


$$A = \tfrac{1}{2} b h = \tfrac{1}{2}(2)(4) = 4$$

$$\therefore \int_0^2 2x\,dx = 4$$

Ex $\displaystyle\int_1^2 x^2\,dx$

We cannot compute the area directly, so we need to approximate using sums or find the limit of the sums to get the exact area



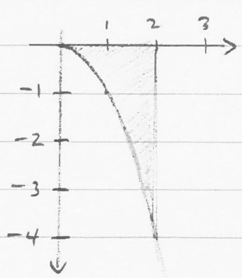
The Definite Integral

We will partition $[1,2]$ into n intervals with $\Delta x = \frac{2-1}{n} = \frac{1}{n}$.

If we take $c_i = 1 + i\Delta x$ then
$$f(x_i) = (1+i\Delta x)^2 = 1 + 2i\Delta x + i^2\Delta x^2 = 1 + \frac{2i}{n} + \frac{i^2}{n^2}$$

$$\int_1^2 x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left(1 + \frac{2i}{n} + \frac{i^2}{n^2}\right)\left(\frac{1}{n}\right)$$
$$= \lim_{n\to\infty} \left[\sum_{i=1}^{n} \frac{1}{n} + \sum_{i=1}^{n} \frac{2i}{n^2} + \sum_{i=1}^{n} \frac{i^2}{n^3}\right]$$
$$= \lim_{n\to\infty} \left[\frac{1}{n}\sum_{i=1}^{n} 1 + \frac{2}{n^2}\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} i^2\right]$$
$$= \lim_{n\to\infty} \left[\frac{1}{n}\cdot n + \frac{2}{n^2}\frac{n(n+1)}{2} + \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$
$$= \lim_{n\to\infty} \left[1 + \frac{n+1}{n} + \frac{2n^2+3n+1}{6n^2}\right]$$
$$= 1 + 1 + \frac{1}{3} = \boxed{7/3}$$

Ex Compute $\int_1^2 -x^2\,dx$



How should we interpret this area?

When we say "area under the curve" we really mean the net area between the curve and the x-axis.

$$\int_1^2 -x^2\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} -(c_i)^2\Delta x$$
$$= -\lim_{n\to\infty} \sum_{i=1}^{n} (c_i)^2\Delta x$$
$$= -\int_1^2 x^2\,dx = -7/3$$

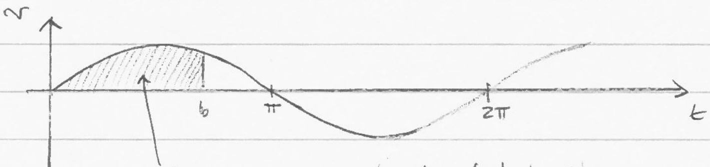
We interpret area below the x-axis as "negative-area". The sign indicates that it is below the axis.

The Definite Integral

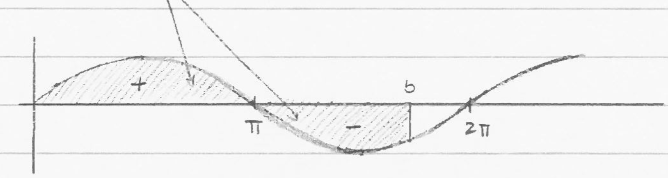
Ex Suppose the velocity of an object is given by $v(t) = \sin t$

$v(t)$ gives velocity at time t

$$\int_0^b v(t)\,dt = \int_0^b \sin t\,dt$$ gives distance travelled from time $t=0$ until time $t=b$.



area corresponds to total change in position



In this case, the velocity is positive on $(0,\pi)$ and negative on $(\pi, b]$ so the position is decreased by the area from π to b.

Ex $\displaystyle\int_0^{2\pi} \sin t\,dt = ?$ Since we have two regions of equal area, one above the axis (+) and one below (−), the value of the integral is 0.

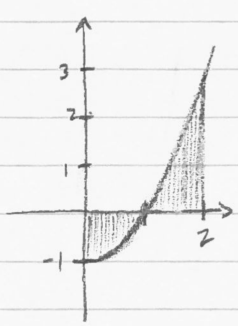
$= 0$

Ex $$\int_0^2 (x^2-1)\,dx = \lim_{n\to\infty} \sum_{i=1}^n \left[\left(\frac{2i}{n}\right)^2 - 1\right]\frac{2}{n}, \qquad \Delta x = \frac{2}{n}, \quad x_i = 0 + i\frac{2}{n} = \frac{2i}{n}$$

$$= \lim_{n\to\infty} \frac{2}{n}\sum_{i=1}^n \left(\frac{4i^2}{n^2} - 1\right)$$

$$= \lim_{n\to\infty} \frac{2}{n}\left[\frac{4}{n^2}\frac{n(n+1)(2n+1)}{6} - n\right]$$

$$= \lim_{n\to\infty} \frac{8(n+1)(2n+1)}{6n^2} - 2 = \frac{16}{6} - 2 = \frac{8}{3} - 2 = \boxed{2/3}$$



The Definite Integral

Theorems

4.1 If f is continuous on $[a,b]$ then f is <u>integrable</u> on $[a,b]$.

4.2 If f and g are integrable on $[a,b]$ then

a) for any constants c and d

$$\int_a^b [c f(x) + d g(x)]\,dx = c\int_a^b f(x)\,dx + d\int_a^b g(x)\,dx$$

b) for any $c \in [a,b]$,

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

4.3 If $g(x) \le f(x)$ for all $x \in [a,b]$ and if f and g are integrable on $[a,b]$ then

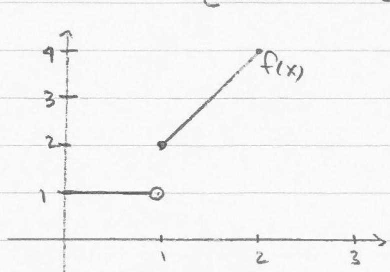
$$\int_a^b g(x)\,dx \le \int_a^b f(x)\,dx$$

Thm 4.1 is based on a "sufficient" but not "necessary" condition for integrability.

$$f(x) \text{ continuous} \Rightarrow \text{integrable}$$

The converse is not true: $f(x)$ being integrable does not imply f is continuous.

Ex Let $f(x) = \begin{cases} 1 & 0 \le x < 1 \\ 2x & 1 \le x \end{cases}$ and find $\int_0^2 f(x)\,dx$



The Definite Integral

Let us assume for the moment that f is integrable. Then

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

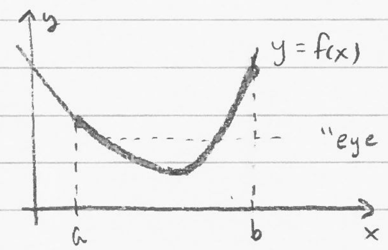
$$= \int_0^1 1\,dx + \int_1^2 2x\,dx$$

$$= \underbrace{1}_{\text{area of square}} + \underbrace{\frac{2+4}{2}\cdot 1}_{\text{area of trapezoid}} = 1+3 = \boxed{4}$$

upper limit on first integral should really be dealt with using $\lim_{b\to 1} \int_0^b f(x)dx$ since the integral is improper.

We will see why we know that <u>piecewise continuous</u> functions like this are integrable next semester.

We can use integrals to compute the average value of a function



What is the average value of $f(x)$ on $[a,b]$?

If we evaluate $f(x)$ at n points taken uniformly from $[a,b]$ then we can estimate the average with

$$\text{Avg} \approx \frac{f(x_1) + f(x_2) + \dots + f(x_n)}{n}$$

or exactly with

$$\text{Avg} = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i)$$

Notice that $\text{Avg} = \lim_{n\to\infty} \frac{1}{(b-a)} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i)$

$$= \frac{1}{b-a} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x \qquad \text{since } \Delta x = \frac{b-a}{n}$$

$$= \frac{1}{b-a}\int_a^b f(x)\,dx$$

$$\boxed{f_{avg} \text{ on } [a,b] = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

See text for the Integral Mean Value Theorem

Ex Suppose a particle has velocity $v(t) = 4 - t^2$ m/s for $t=1$ to $t=2$. How far does it travel, and what is its avg velocity?

The particle's displacement is given by $\int_1^2 \underbrace{(4-t^2)}_{m/s}\, \underbrace{dt}_{s} \to m$

$$\int_1^2 (4-t^2)\,dt = \int_1^2 4\,dt - \int_1^2 t^2\,dt$$

$$= 4 \cdot 1 - 7/3$$

rect. of height 4 and width 1.

since we already know $\int_1^2 x^2\,dx = 7/3$

$$= 4 - 7/3 = \frac{12-7}{3} = 5/3 \to \boxed{\text{particle traves } 5/3 \text{ m.}}$$

The average velocity is $\underbrace{\frac{1}{2-1}}_{1/sec} \underbrace{\int_1^2 (4-t^2)\,dt}_{m} = \frac{1}{1}\cdot\frac{5}{3} = \boxed{\frac{5}{3} \text{ m/sec}}$