Derivatives of Inverse Trigonometric Functions

Find $\frac{d}{dx}\sin^{-1}x$.

Implicit differentiation is the key. Let $y = \sin^{-1}x$
so

$$\sin y = x \qquad -\pi/2 \le y \le \pi/2$$

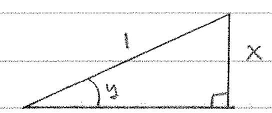
Differentiate

$$\frac{d}{dx}\sin y = \frac{d}{dx}x$$

$$\cos y \, y' = 1$$

$$y' = 1/\cos y \qquad \text{only defined for } -\pi/2 < y < \pi/2$$

Since $y' = \frac{d}{dx}\sin^{-1}x$ is our objective but y is a variable we introduced, we need to replace $\cos y$ with an equivalent function of x.



From our original substitution $x = \sin y$:

The adjacent side is $\sqrt{1-x^2}$ so $\cos y = 1/\sqrt{1-x^2}$

Therefore $\boxed{\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}}$ $\quad -1 < x < 1$

Find $\frac{d}{dx}\tan^{-1}x$. Let $y = \tan^{-1}x$ so $x = \tan y$, $-\pi/2 < y < \pi/2$

$$\frac{d}{dx}\tan y = \frac{d}{dx}x$$

$$\sec^2 y \, y' = 1 \Rightarrow y' = 1/\sec^2 y$$

We could use $1/\sec^2 y = \cos^2 y$, but it's more helpful to use the identity

$$\sin^2\theta + \cos^2\theta = 1$$

$$\Rightarrow \tan^2\theta + 1 = \sec^2\theta$$

So $y' = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$ $\qquad \boxed{\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}}$

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Similarly, we find $\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$, $-1 < x < 1$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

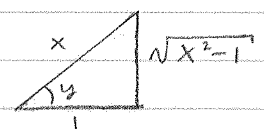
$$\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}, \quad |x|>1$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{|x|\sqrt{x^2-1}}, \quad |x|>1$$

Proof of $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$, $|x|>1$

If $y = \sec^{-1}x$ then $y \in [0, \pi/2) \cup (\pi/2, \pi]$ and $|x| \geq 1$

$x = \sec y$

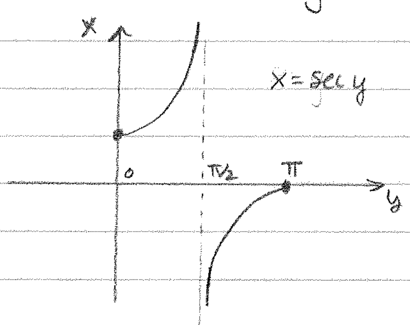


$$\frac{d}{dx}\sec y = \sec y \tan y \frac{dy}{dx} = \frac{d}{dx}x = 1$$

$$\therefore \frac{dy}{dx} = \frac{1}{\sec y \tan y}$$

When $y \in [0, \pi/2)$ we have $\sec y \geq 1$ so $x \geq 1$, $\tan y \geq 0$

When $y \in (\pi/2, \pi]$ we have $\sec y \leq -1$ so $x \leq -1$, $\tan y \leq 0$



So, $\sec y = x$, $\tan y = \pm\sqrt{x^2-1}$

When $x \geq 1$, $\tan y = \sqrt{x^2-1}$

When $x \leq -1$, $\tan y = -\sqrt{x^2-1}$

So
$$\frac{d}{dx}\sec^{-1}x = \begin{cases} \frac{1}{x\sqrt{x^2-1}} & \text{if } x>1 \\ \frac{-1}{x\sqrt{x^2-1}} & \text{if } x<1 \end{cases}$$

($x=1$ and $x=-1$ are not possible since then $\tan y = 0$ and the ratio is not defined)

but this is equivalent to $\frac{d}{dx}\sec^{-1}x = \frac{1}{|x|\sqrt{x^2-1}}$