Exponential and Logarithmic Functions

Question: What behavior characterizes a linear function $f(x)$?

Answer: Increasing x by Δx will change $y = f(x)$ by a fixed amount Δy.

Ex

x	0	1	2	3	4	5	6
y	1	3	5	7	9	11	13

as x changes by $\Delta x = 1$, y changes by $\Delta y = 2$

Now consider the function given by

x	0	1	2	3	4	5	6
y	1	2	4	8	16	32	64

Here as x increases by $\Delta x = 1$, y changes by different amounts → what is the pattern?

Every time x increases by 1, y doubles.

The function is $\boxed{y = f(x) = 2^x}$.

If a number b exists such that $f(x+1) = b f(x)$ then f is an <u>exponential function</u> and

$$f(x) = b^x$$

b is the <u>base</u> and
x is the <u>exponent</u>

Ex $P(t) = 103(1/2)^t$ is an exponential function that decreases by a factor of $1/2$ (50%) each time t increases by 1.

Ex $y = \sqrt{x} = x^{1/2}$ is <u>not</u> an exponential function since the exponent is a fixed value.

Exponential and Logarithmic Functions

How do we compute expressions like 2^3 or 8.1^5?

How about $5^{2/3}$?

$$5^{2/3} = (5^2)^{1/3} = \sqrt[3]{25}$$

What about 2^x if x is irrational?

Rules of Exponents

1. For integers m and n: $x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$
2. For any real number p: $x^{-p} = 1/x^p$
3. For any real numbers p & q: $(x^p)^q = x^{pq}$
4. For any real numbers p & q: $x^p \cdot x^q = x^{p+q}$

Ex As x increases which of the following functions grows most quickly?

$$f(x) = x^{10}, \quad g(x) = 10^x$$

x	f	g
0	0	1
5	9,765,625	100,000
10	10,000,000,000	10,000,000,000
15	576,650,390,625	1,000,000,000,000,000
20	10,240,000,000,000	100,000,000,000,000,000,000

It appears that $g(x) = 10^x$ grows most quickly when x gets larger, even though f(x) grows more quickly when x is small.

We will soon be able to prove that an increasing exponential function

$$b^x \quad \text{with } b > 1$$

will (eventually) grow more quickly than any power function

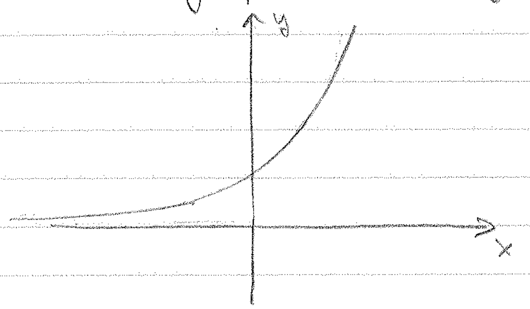
$$x^a$$

for any real number a.

The natural base for exponential functions is the irrational number 2.718281828 46..., designated by e

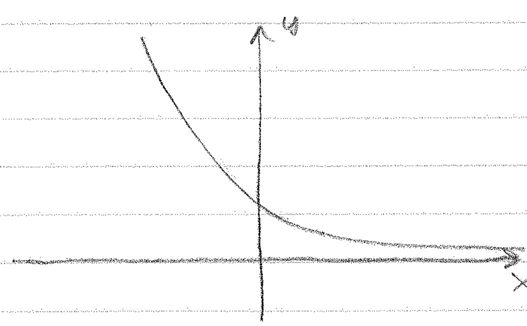
There are several way to compute e but all of them require mathematics we've not yet seen. It turns out however, that it is convenient to use e as our base.

When we graph the exponential function b^x we will find graphs like Figures 0.69-0.71.



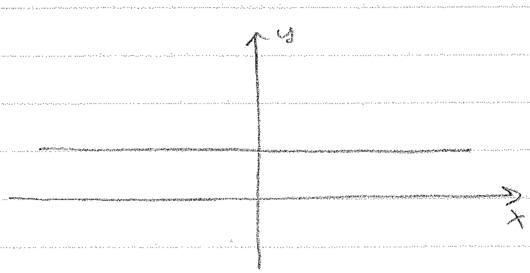
The graph will look like this if $b > 1$

Exponential Growth



This is what the graph will look like if $b < 1$

Exponential decay



When $b = 1$ the function is constant

$1^x = 1$ for all x.

Exponential and Logarithmic Functions

From these graphs we see that as long as $b \neq 0$ then $f(x) = b^x$ is an invertible function.

We define the inverse of b^x to be $\log_b x$.

<u>Def</u> For an positive number $b \neq 1$, the <u>logarithm</u> function with base b, written

$$\log_b x$$

is defined by

$$y = \log_b x \quad \text{if and only if} \quad x = b^y.$$

Two frequently used logarithms are the <u>common logarithm</u>, $\log_{10} x$ or $\log x$, and the <u>natural logarithm</u>, $\ln x$.

<u>Ex</u> Without using a calculator, evaluate

$$\log(100), \quad \log(1/10), \quad \ln e^5$$

$$\log(100) = \log(10^2) = 2$$

$$\log(1/10) = \log(10^{-1}) = -1$$

$$\ln e^5 = 5$$

Since $\ln x$ is the inverse of e^x, we know e^x is the inverse of $\ln x$

<u>Ex</u> Solve $\ln(2x+1) = 4$

$$e^{\ln(2x+1)} = e^4$$

$$2x + 1 = e^4$$

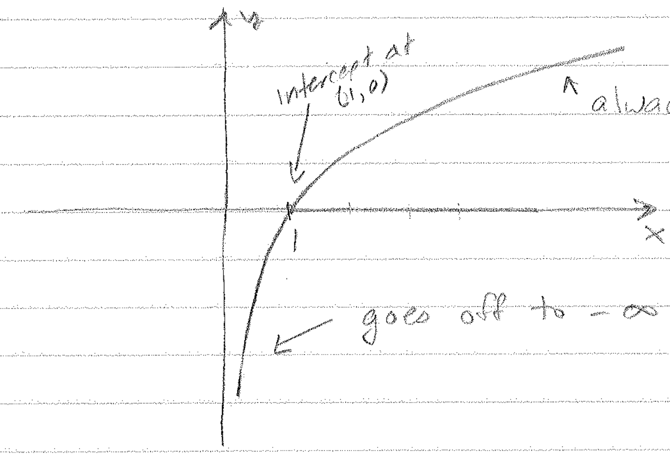
$$2x = e^4 - 1$$

$$\boxed{x = \tfrac{1}{2}(e^4 - 1)}$$

Exponential and Logarithmic Functions

To graph $\log_b x$ we only need to invert the graph of b^x.

When $b > 1$, the graph of $\log_b x$ looks like (see figure 0.72)



Thm For any positive base $b \neq 1$

1. $\log_b x$ is defined only for $x > 0$,
2. $\log_b 1 = 0$, and
3. if $b > 1$ then $\log_b x < 0$ for $0 < x < 1$ and $\log_b x > 0$ for $x > 1$.

Thm – Properties of Logarithms

For any positive base $b \neq 1$ and positive numbers x and y, we have

1. $\log_b (xy) = \log_b x + \log_b y$
2. $\log_b (x/y) = \log_b x - \log_b y$
3. $\log_b (x^y) = y \log_b x$

Ex Simplify $\log(100\cdot 10^x)$, $\ln 5e^x$, $\log(64/2^3)$

$$\log(100\cdot 10^x) = \log(100) + \log 10^x =$$
$$= \log 10^2 + \log 10^x$$
$$= 2\log 10 + x\log 10$$
$$= 2\cdot 1 + x\cdot 1 = \boxed{2+x}$$

$$\ln 5e^x = \ln 5 + \ln e^x$$
$$= \ln 5 + x\ln e$$
$$= \ln 5 + x\cdot 1 = \boxed{\ln 5 + x}$$

$$\log(64/2^3) = \log 64 - \log 2^3$$
$$= \log 2^6 - 3\log 2$$
$$= 6\log 2 - 3\log 2$$
$$= 3\log 2 = \log 2^3 = \boxed{\log 8}$$

Ex Find a and b so that $f(x) = ae^{bx}$ passes through $(0,3)$ and $(2,7)$.

$$3 = ae^{b\cdot 0} = ae^0 = a \qquad \boxed{a=3}$$

so

$$7 = 3e^{2b} \longrightarrow 7/3 = e^{2b}$$

$$\ln 7/3 = \ln e^{2b} = 2b\ln e = 2b$$

$$\boxed{b = \tfrac{1}{2}\ln 7/3}$$

Suppose one wants to compute $\log_a x$ for some base a but only has a calculator that computes $\log_b x$ (e.g. $\log_{10} x$).

We want to compute $y = \log_a x$. This means $a^y = x$. So

$$a^y = x$$

$$\log_b a^y = \log_b x$$

$$y \log_b a = \log_b x$$

$$y = \log_b x / \log_b a$$

$$\boxed{\log_a x = \log_b x / \log_b a}$$

Ex. Compute $\log_2 1000$.

$$\log_2 1000 = \log 1000 / \log 2$$

$$\approx 3/0.30103$$

$$\simeq 9.9658$$

Note that $2^{10} = 1024$ and $2^9 = 512$, so this answer makes sense.

Interestingly, it turns out that the exponential functions e^x and e^{-x} often appear together in expressions. It becomes convenient to define functions with $e^x + e^{-x}$ and $e^x - e^{-x}$. For reasons that will become apparent later, we define

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \text{Hyperbolic cosine.}$$

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \text{Hyperbolic sine.}$$

Like $\cos x$, $\cosh x$ is an even function. This function describes both the shape of a cable suspended at its two ends and the shape of a freestanding arch.