The Fundamental Theorem of Calculus

Amazingly there is a direct connection between the antiderivative of $f(x)$ and the area under the curve of $f(x)$.

The Fundamental Theorem of Calculus - Part 1

If f is continuous on $[a,b]$ and $F(x)$ is any antiderivative of $f(x)$ then
$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof

We can partition $[a,b]$ into n equal sized subintervals $[x_{i-1}, x_i]$ with
$$a = x_0 < x_1 < x_2 < \dots < x_n = b \quad \text{and} \quad \Delta x = x_i - x_{i-1}$$

So
$$\begin{aligned} F(b) - F(a) &= F(x_n) - F(x_0) \\ &= F(x_n) - F(x_{n-1}) + F(x_{n-1}) \\ &\quad - F(x_{n-2}) + F(x_{n-2}) \\ &\quad \dots - F(x_1) + F(x_1) - F(x_0) \\ &= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \end{aligned}$$

From the MVT we know $f(c_i) = F'(c_i) = \dfrac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}$ for some $c_i \in (x_{i-1}, x_i)$.

$$\therefore \quad f(c_i)\Delta x = F(x_i) - F(x_{i-1}) \quad \text{for some } c_i \in [x_{i-1}, x_i]$$

So
$$F(b) - F(a) = \sum_{i=1}^{n} f(c_i)\Delta x$$

Taking the limit as $n \to \infty$ on both sides we have

$$F(b) - F(a) = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x = \int_a^b f(x)\,dx \qquad \checkmark$$

This is rather amazing! The area under $f(x)$ on $[a,b]$ is completely and exactly determined by the difference of any one of it's antiderivatives at $x=b$ and $x=a$.

The Fundamental Theorem of Calculus

Ex Last class we showed $\int_0^2 2x\,dx = 4$

$$\int_0^2 2x\,dx = x^2\Big|_0^2 = 2^2 - 0^2 = \boxed{4}$$

Here we have used the notation $F(x)\Big|_a^b = F(b) - F(a)$.

Ex $\int_1^2 x^2\,dx = \frac{x^3}{3}\Big|_1^2 = \frac{2^3}{3} - \frac{1^3}{3} = \frac{8}{3} - \frac{1}{3} = \boxed{\frac{7}{3}}$ ✓

Ex $\int_0^\pi \sin x\,dx = -\cos x\Big|_0^\pi = -\cos\pi - (-\cos 0) = -(-1) - (-1) = \boxed{2}$

Ex $\int_1^x (3t^2+1)\,dt$ Here the upper limit is the variable x.

$$\int_1^x (3t^2+1)\,dt = t^3 + t\Big|_1^x = x^3 + x - (1^3+1) = x^3 + x - 2$$

Notice that this is a function of x, so $F(x) = \int_1^x (3t^2+1)\,dt$.
Notice further that $\frac{d}{dx}\int_1^x (3t^2+1)\,dt = \frac{d}{dx}(x^3+x-2) = 3x^2+1$

which is the original integrand with x as the variable.

The Fundamental Theorem of Calculus - Part 2.

If f is continuous on $[a,b]$ then

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

Ex Find the equation of the line tangent to

$$y = \int_1^x \sqrt{t^2+1}\,dt \quad \text{at} \quad x = 1.$$

As usual, we need the point on the curve corresponding to $x=1$ and the slope at $x=1$.

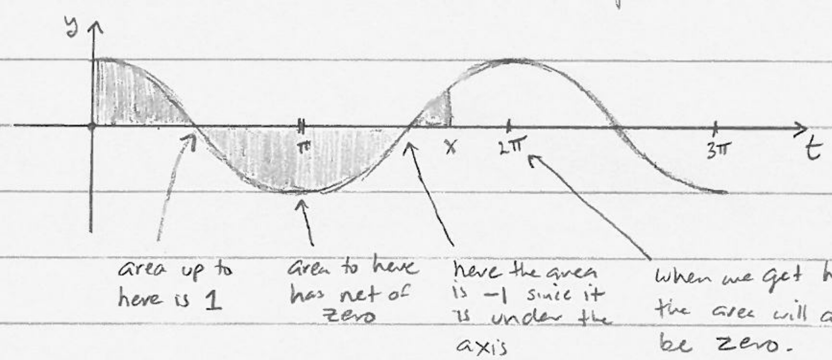
At $x=1$ $\quad y = \int_1^1 \sqrt{t^2+1}\,dt = 0$ since the interval has length zero. Our point is $(1,0)$

$y' = \sqrt{x^2+1}$ by FTC part 2 so at $x=1$, $y' = \sqrt{2}$

So

$$y - 0 = \sqrt{2}\,(x-1) \rightarrow \boxed{y = \sqrt{2}\,(x-1)}$$

Ex Graph $\cos(t)$ from $t=0$ up to $t=x$. Observe that the area under the curve corresponds to $\sin x$.



The pattern described above is exactly that for $\sin x$.