Implicit Differentiation

Compute the following derivatives

$\frac{d}{dx} x^2 = 2x$

$\frac{d}{dx} (\sin x)^2 = 2(\sin x)^1 \cdot \cos x = 2 \sin x \cos x$

$\frac{d}{dx} (f(x))^2 = 2 f(x) \cdot f'(x)$

$\frac{d}{dx} y^2 = 2y \cdot y' = 2y \frac{dy}{dx}$

$\frac{d}{dx} (\sin y)^2 = 2 \sin y \cdot \cos y \cdot y' = 2 \sin y \cos y \frac{dy}{dx}$

Notice that when we are taking the derivative of the variable y with respect to x we "finish" the chain rule with dy/dx or y'.

We could do this in the first examples as well, but $dx/dx = 1$ so there is no need.

Taking the derivative of a function of y wrt x is called <u>Implicit Differentiation</u>.

Ex. Find the equation of the line tangent to $x^2+y^2=25$ at the point $(3,4)$.

We could solve for y and compute y', but this is messy.

Alternatively, we can differentiate both sides wrt x (why x?) and solve for y'.

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(25)$$

$$2x + 2yy' = 0$$

$$2yy' = -2x$$

$$y' = -x/y$$

Implicit Differentiation

Once we have y', we can find the slope at $(3,4)$.

$$m = y'\Big|_{(3,4)} = -3/4$$

We now have a point and a slope, so

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -\frac{3}{4}(x - 3)$$

$$y = -\frac{3}{4}x + \frac{9}{4} + 4 = -\frac{3}{4}x + \frac{25}{4}$$

$$\boxed{y = -\frac{3}{4}x + \frac{25}{4}}$$

When we differentiate implicitly, we

1. will need to use the chain rule and end derivatives wrt a different variable with an explicit derivative

2. will often finish with an implicit expression containing both the independent and dependent variables

3. Will often need to use algebraic manipulation to Solve for the derivative of the dependent variable.

Implicit Differeniation is straight forward if you

1. use the chain rule carefully

2. use algebra correctly.

Implicit Differentiation

Ex. Find $y'(x)$ if $3xy + y^2 = 2x$

$$\frac{d}{dx}\left(3xy + y^2\right) = \frac{d}{dx}(2x)$$

$$\frac{d}{dx}(3x)(y) + \frac{d}{dx}y^2 = 2$$

$$3y + 3xy' + 2yy' = 2$$

Now solve for y'

$$3xy' + 2yy' = 2 - 3y$$

$$y'(3x+2y) = 2-3y$$

$$\boxed{y' = (2-3y)/(3x+2y)}$$

Do not try to solve explicitly in terms of x alone – it's okay to have both x and y in the answer.

You may find it helpful to always end each derivative with x' or y', but then recognize that $x' = \frac{dx}{dx} = 1$.

Ex Find y' if $3xy^2 + xy = y + x$

$$\frac{d}{dx}\left(3xy^2 + xy\right) = \frac{d}{dx}(y+x)$$

$$3x'y^2 + 3x2yy' + x'y + xy' = y' + x'$$

$$3y^2 + 6xyy' + y + xy' = y' + 1$$

$$6xyy' + xy' - y' = 1 - 3y^2 - y$$

$$y'(6xy + x - 1) = 1 - 3y^2 - y$$

$$\boxed{y' = \frac{1-3y^2-y}{6xy+x-1}}$$

Implicit Differentiation

Find y'' if $2xy + x = y$

$$\frac{d}{dx}(2xy + x) = \frac{d}{dx} y$$

$$2y + 2xy' + 1 = y' \qquad \text{so} \qquad y' = \frac{2y+1}{1-2x}$$

$$\frac{d}{dx}(2y + 2xy' + 1) = \frac{d}{dx} y'$$

$$2y' + 2y' + 2xy'' = y''$$

$$y'' = \frac{4y'}{1-2x} = \frac{4}{1-2x} \frac{2y+1}{1-2x} = \frac{8y+4}{(1-2x)^2}$$

$$\boxed{y'' = \frac{8y+4}{(1-2x)^2}}$$