Increasing and Decreasing Functions

We have already begun to use information from the derivative $f'(x)$ to understand the behavior of $f(x)$. Now we want to organize and extend these ideas

Def. A function f is (strictly) increasing on an interval I if, for every $x_1, x_2 \in I$ with $x_1 < x_2$, $f(x_1) < f(x_2)$

A function f is (strictly) decreasing (given the same conditions above) if $f(x_1) > f(x_2)$.

Thm Suppose that f is differentiable on an interval I.

1. If $f'(x) > 0$ for all $x \in I$ then f is increasing on I
2. If $f'(x) < 0$ for all $x \in I$ then f is decreasing on I.

Proof

Choose any two points on I and call them x_1 and x_2 with $x_1 < x_2$. Suppose we are trying to prove 1. By the MVT we know $c \in (x_1, x_2)$ exists so that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - x_1} > 0$$

Since $x_2 - x_1 > 0$ we can multiply both sides of the inequality by it to get

$$f(x_2) - f(x_1) > 0$$

where the direction of the inequality is unchanged.
So

$$f(x_2) > f(x_1)$$

and so the function is increasing.

A similar argument holds for 2.

Increasing and Decreasing Functions

<u>Ex</u> Find the intervals on which $f(x) = x^3 - 3x^2 - 24x + 5$ is increasing and those on which it is decreasing.

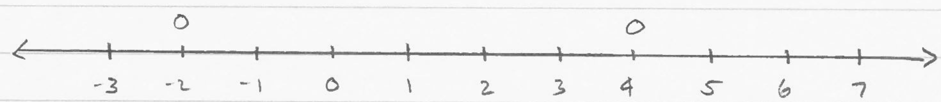
$$f(x) = x^3 - 3x^2 - 24x + 5$$

$$f'(x) = 3x^2 - 6x - 24$$

By the intermediate value theorem, we know that if $f'(x)$ changes sign in an interval it must have a zero in that interval — this suggests we find the zeros of $f'(x)$

$$3x^2 - 6x - 24 = 0$$
$$(3x + 6)(x - 4) = 0$$
$$3x = -6 \quad \text{or} \quad x = 4$$
$$x = -2 \quad \text{or} \quad x = 4$$

$f'(x)$



Now we examine values of x from 3 intervals

$$-\infty < x < -2$$
$$-2 < x < 4$$
$$4 < x < \infty$$

$$f'(-3) = 3(-3)^2 - 6(-3) - 24 = 27 + 18 - 24 = 21 > 0$$

So $f(x)$ is increasing on $-\infty < x < -2$

$$f'(0) = 3(0)^2 - 6(0) - 24 = -24 < 0 \quad \text{so}$$

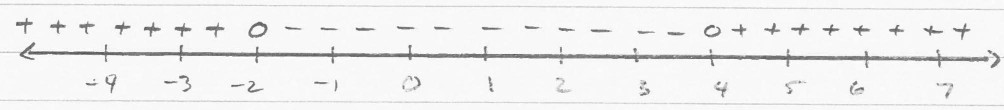
$f(x)$ is decreasing on $-2 < x < 4$

$$f'(5) = 3(5)^2 - 6(5) - 24 = 75 - 30 - 24 = 21 > 0$$

So $f(x)$ is increasing on $4 < x < \infty$

Increasing and Decreasing Functions

We can diagram our results as



where the + and − signs indicate increasing or decreasing for f and the sign of f'.

From the above diagram we can see that a local maximum will occur at $x = -2$ and a local minimum will occur at $x = 4$.

Thm (First derivative test)

Suppose that f is continuous on $[a, b]$ and $c \in (a, b)$ is a critical number.

1) if $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b) then $f(c)$ is a local maximum.

2) if $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b) then $f(c)$ is a local minimum.

3) if $f'(x)$ has the same sign on (a, c) and (c, b) then $f(c)$ is not a local extremum.

Ex Find the extrema of $f(x) = 3x^4 + 40x^3 - 0.06x^2 - 1.2x$ and sketch a graph of the function.

We first look for critical numbers.

$$f'(x) = 12x^3 + 120x^2 - 0.12x - 1.2$$

$$= 12x^2(x + 10) - 0.12(x + 10)$$
$$= (12x^2 - 0.12)(x + 10)$$
$$= 12(x^2 - 0.01)(x + 10) = 12(x - 0.1)(x + 0.1)(x + 10)$$

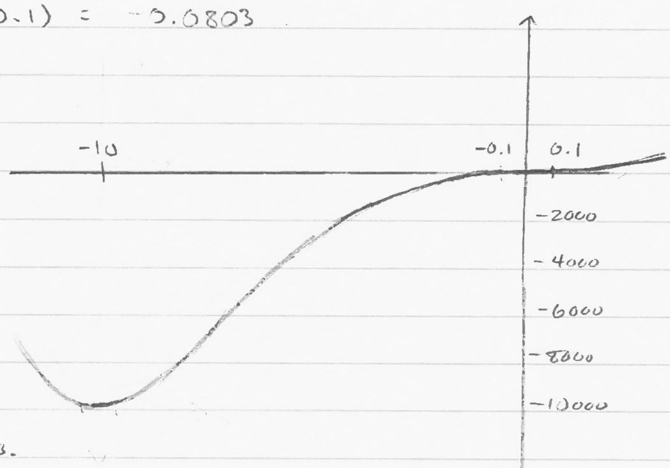
Diagramming we find

Local min at $x = -10$

Local max at $x = -0.1$

Local min at $x = 0.1$

$$f(-10) = -9994$$
$$f(-0.1) = 0.07970$$
$$f(0.1) = -0.0803$$



Notice how poorly the graph shows the behavior near two of the critical numbers.

A quick graph on your calculator might lead you to miss the fact the function is decreasing between $x = -0.1$ and $x = 0.1$.

Ex Sketch a graph of a continuous function with the given properties.

$f(1) = 2$, $f(2) = 0$, $f'(x) < 0$ for $x < 0$ and $1 < x < 3$, $f'(x) > 0$ for $0 < x < 1$, and $f'(x) = 0$ for $3 < x$