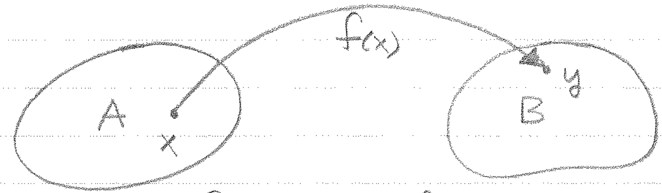
Inverse Functions



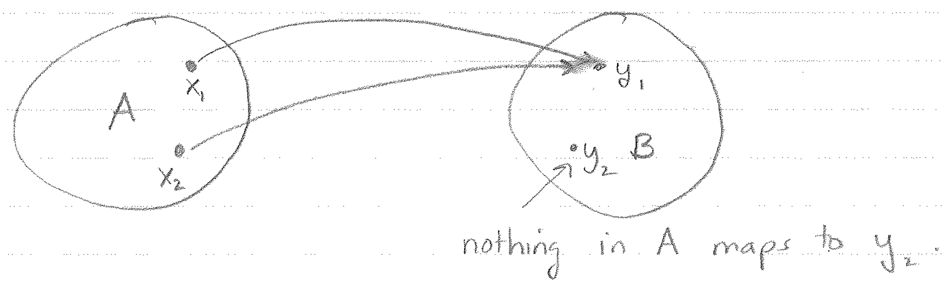
$f(x)$ is a function that maps set A (the domain of f) to the set B.

Because of how functions are defined we know that each x in A maps to only one y in B.

It may or may not, however, be the case that a given y in B is mapped from only one x in A.

It is also possible that a y in B is not mapped from any x in A — showing that B may not be the range of f.

Sometimes, however, it is possible to do this reverse mapping. What property must f have to make this possible?



What the question is asking really is — "is there a function f^{-1} whose domain is B and whose range is A such that if $y = f(x)$ then $x = f^{-1}(y)$?

Def A function f is called <u>one-to-one</u> when for every $y \in \text{Range}\{f\}$, there is exactly one $x \in \text{Domain}\{f\}$ for which $y = f(x)$.

The phrase "there is exactly one" means
1) there is at least one, and
2) there is no more than one

Inverse Functions

Ex The function $y = x^3$ is one-to-one since for any y we can tell the value of x that gives it.

The function $f(x) = x^3$ is <u>invertible</u> and $x = f^{-1}(y)$ since

$$x = y^{1/3}$$

The domain of $f(x) = x^3$ is $\mathbb{R}$ and the range is also $\mathbb{R}$. So, the domain and range of $f^{-1}(x) = x^{1/3}$ are also both $\mathbb{R}$.

Def The function $f(x)$ is <u>invertible</u> and has inverse $f^{-1}(x)$ if $f(x)$ is one-to-one

Thm A function f has an inverse iff (if and only if) it is one-to-one.

Ex Does $f(x) = x^2$ have an inverse?

$f(1) = 1, \quad f(2) = 4, \quad f(3) = 9, \quad$ etc.

Can we say $f^{-1}(x) = \sqrt{x}$? Maybe...

Whether or not we consider $f(x) = x^2$ to have an inverse depends on the domain and range.

If the domain is $\mathbb{R}$ then no, $f(x)$ <u>does not</u> have an inverse since it is not one-to-one.

$$f(-2) = f(2) = 4$$

If, however, we take the domain and range to be all non-negative reals

$$\{x \in \mathbb{R} \mid x \geq 0\}$$

then $f(x)$ is one-to-one and is invertible.

This is called <u>restricting the domain</u>.

Ex Find the inverse of $f(x) = 2x^3+5$

Start with $y = f(x) = 2x^3+5$ and solve for x.

$$y = 2x^3+5$$
$$y-5 = 2x^3$$
$$\tfrac{1}{2}(y-5) = x^3$$
$$\left[\tfrac{1}{2}(y-5)\right]^{1/3} = x$$

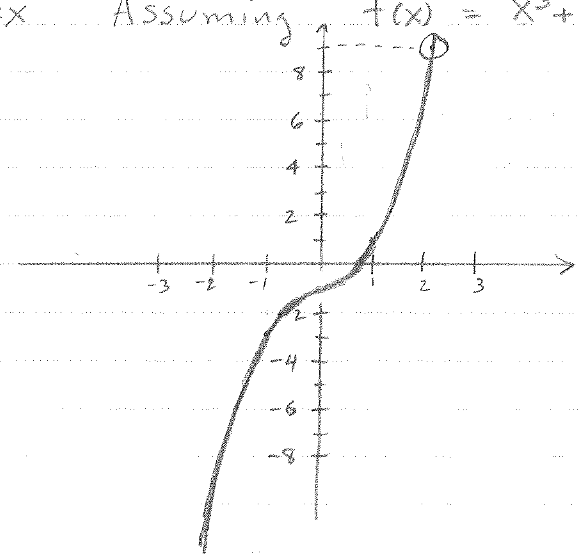
So $x = f^{-1}(y) = \left[\tfrac{1}{2}(y-5)\right]^{1/3}$

Since we usually use x as the independent variable and since it is just a placeholder (dummy variable) we can switch x and y to get

$$y = f^{-1}(x) = \left[\tfrac{1}{2}(x-5)\right]^{1/3} = \sqrt[3]{\tfrac{1}{2}(x-5)}$$

$$\boxed{f^{-1}(x) = \sqrt[3]{\tfrac{1}{2}(x-5)}}$$

Ex Assuming $f(x) = x^3+x-1$ is invertible, what is $f^{-1}(9)$?



We can graph f(x) and see that it does appear to be one-to-one.

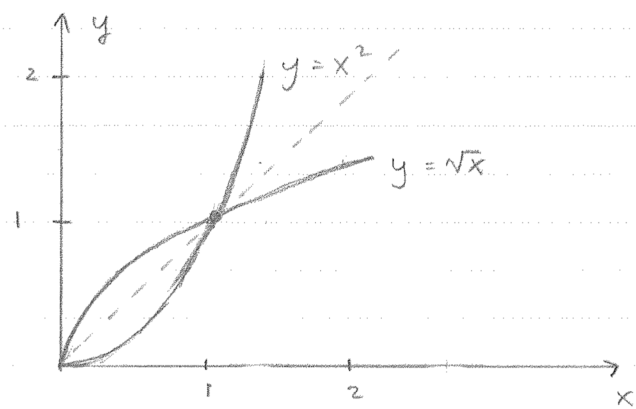
By inspection it appears that $f(2) = 9$. Checking, we find

$$2^3+2-1 = 8+2-1 = 9 \checkmark$$

So $\boxed{f^{-1}(9) = 2.}$

Inverse Functions

When the domain is the nonnegative reals, $f(x) = x^2$ is invertible. The graph of x^2 and its inverse $\sqrt{x}$ is shown below



Notice that each of these curves is the reflection of the other about the line $y = x$.

Since $y = f(x)$, $x = f^{-1}(y)$ and if we swap the roles of x and y we have
$$y = f^{-1}(x).$$

Swapping x and y amounts to "flipping" the axes.