Limits Involving Infinity; Asymptotes

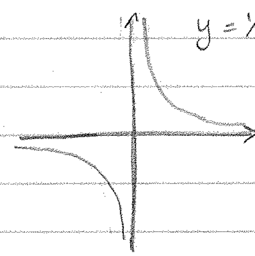
We have seen that some limits tend to infinity

$$\lim_{x\to 0} \frac{1}{x}, \qquad \lim_{x\to \pi/2} \sec x$$

Remember, infinity is not a number!

We do, however, use the symbol ∞ in many places where confusion can arise.

For example, $\lim_{x\to 0+} 1/x$ does not exist. We know from the graph that the limit fails to exist because the function goes to infinity. We can write



$$\lim_{x\to 0^+} 1/x = \infty$$

or

$$\lim_{x\to 0^+} 1/x = +\infty$$

Similarly

$$\lim_{x\to 0^-} 1/x = -\infty$$

Ex $\lim_{x\to 0^+} 1/x^2$ $\quad$ x^2 goes to 0 from above so $1/x^2$ will go to $+\infty$ $\rightarrow$ $\lim_{x\to 0^+} 1/x^2 = \infty$

$\lim_{x\to 0^-} 1/x^2$ $\quad$ x^2 goes to 0 from above so $1/x^2$ will go to $+\infty$ $\rightarrow$ $\lim_{x\to 0^-} 1/x^2 = \infty$

Ex $\lim_{x\to 3^+} \frac{x}{(x-3)^3}$ $\quad$ As $x\to 3^+$, $(x-3)^3 > 0$ and approaches 0.

$\therefore$ $1/(x-3)^3 \to +\infty$

$\therefore$ $x/(x-3)^3 \to +\infty$ $\quad$ So $\lim_{x\to 3^+} \frac{x}{(x-3)^3} = \infty$

$\lim_{x\to 3^-} \frac{x}{(x-3)^3}$ $\quad$ As $x\to 3^-$, $(x-3)^3 < 0$ since $x-3<0$, $x-3\to 0^-$

So $(x-3)^3 \to 0^-$.

$\therefore$ $\lim_{x\to 3^-} \frac{x}{(x-3)^3} = -\infty$

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Another way that limits can involve infinity is because we want the limit as $x \to \infty$ or $x \to -\infty$

Ex $\lim_{x\to\infty} \frac{1}{x}$ as $x \to \infty$, $\frac{1}{x} \to 0$ so $\boxed{\lim_{x\to\infty} \frac{1}{x} = 0}$

$\lim_{x\to-\infty} \frac{1}{x}$ as $x \to -\infty$ $\frac{1}{x} \to 0$ from below. $\boxed{\lim_{x\to-\infty} \frac{1}{x} = 0}$

These limits show that $1/x$ has a <u>horizontal asymptote</u> of $y=0$.

Ex $\lim_{x\to\infty} \frac{5x+2}{x} = \lim_{x\to\infty}\left(\frac{5x}{x} + \frac{2}{x}\right) = \lim_{x\to\infty}\left(5 + \frac{2}{x}\right) = 5 + \lim_{x\to\infty} \frac{2}{x}$

$= 5 + 0 = 5$

So $y=5$ is a horizontal asymptote

Thm: For any rational $t > 0$ $\lim_{x\to\infty} 1/x^t = 0$

and if $t = p/q$, q odd then $\lim_{x\to-\infty} 1/x^t = 0$

Ex $\lim_{x\to\infty} 1/x^{\sqrt{2}} = 0$. This makes sense because as $x \to \infty$, $x^{\sqrt{2}} \to \infty$

Ex $\lim_{x\to\infty} x^2+2 = \infty$ $\qquad$ $\lim_{x\to\infty} -2x+1 = -\infty$

$\lim_{x\to\infty} 3x^2+5x-1 = \infty$ $\qquad$ $\lim_{x\to\infty} -5x^3+2x^2+1 = -\infty$

$\lim_{x\to\infty} 16x^3-x^2+2x-5 = \infty$ $\qquad$ $\lim_{x\to\infty} -x^2+1 = -\infty$

Thm: For any polynomial $p(x)$ of degree $n>0$,

$$\lim_{x\to\infty} p(x) = \begin{cases} \infty & \text{if } a_n > 0 \\ -\infty & \text{if } a_n < 0 \end{cases}$$

where

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

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Some limits of rational funtions are 0, some are constant, and some are infinite

Suppose $p(x) = a_n x^n + \cdots + a_0$ and $q(x) = b_m x^m + \cdots + b_0$
Then

$$\lim_{x\to\infty} p(x)/q(x) = \begin{cases} 0 & \text{if } n<m \\ a_n/b_m & \text{if } n=m \\ \pm\infty & \text{if } n>m \quad (\text{sign is same as } a_n/b_m) \end{cases}$$

Ex $\displaystyle\lim_{x\to\infty} \frac{3x^2+2}{5x^4+3x+1}$ This limit appears to be ∞/∞ which is an indeterminate form.

We can simplify it by dividing both the numerator and denominator by the largest power of x in the denominator.

$$\frac{3x^2+2}{5x^4+3x+1}\left(\frac{1/x^4}{1/x^4}\right) = \frac{3/x^2 + 2/x^4}{5 + 3/x^3 + 1/x^4}$$

$$\lim_{x\to\infty} \frac{3/x^2 + 2/x^4}{5 + 3/x^3 + 1/x^4} = \frac{0+0}{5+0+0} = \frac{0}{5} = 0.$$

Ex $\displaystyle\lim_{x\to\infty} \frac{5x^3+2x^2}{1-x^3} = \lim_{x\to\infty} \frac{5x^3+2x^2}{1-x^3}\left(\frac{1/x^3}{1/x^3}\right)$

$$= \lim_{x\to\infty} \frac{5+2/x}{1/x^3-1} = \frac{5+0}{0-1} = \frac{5}{-1} = -5$$

Ex $\displaystyle\lim_{x\to\infty} \frac{3x^3+x^2+3x+2}{x^2+1} = \lim_{x\to\infty} \frac{3x^3+x^2+3x+2}{x^2+1}\left(\frac{1/x^2}{1/x^2}\right)$

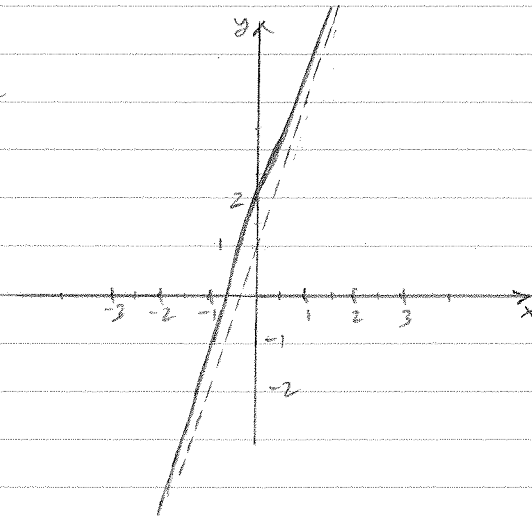
$$= \lim_{x\to\infty} \frac{3x+1+3/x+2/x^2}{1+1/x^2}$$

$$= \frac{\infty}{0} = \infty$$

In cases where the degree of the numerator is one greater than the degree of the denominator There will be an <u>oblique asymptote</u> or <u>slant asymptotes</u>.

Limits Involving Infinity; Asymptotes

The graph of $\dfrac{3x^3 + x^2 + 3x + 2}{x^2+1}$ looks like



We can find the equation of the asymptote by long division

$$
\begin{array}{r|llll}
 & 3x & +\ 1 & & \\
\cline{2-5}
x^2+1 & 3x^3 + & x^2 & +\ 3x & +\ 2 \\
- & 3x^3 & & +3x & \\
\cline{2-4}
 & 0 & x^2 & 0 & +\ 2 \\
 & - & x^2 & & +\ 1 \\
\cline{3-5}
 & & 0 & & 1 \leftarrow \text{remainder}
\end{array}
$$

So $\dfrac{3x^2 + x^2 + 3x + 2}{x^2+1} = 3x + 1 + \dfrac{1}{x^2+1}$

As $x \to \infty$ $\dfrac{1}{x^2+1} \to 0$ so the $3x+1$ is dominant (the significant part)

$y = 3x+1$ is the equation of the asymptote.