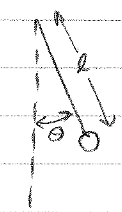
Linear Approximations and Newton's Method

When the motion of a pendulum is studied in elementary physics the differential equation

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0$$



arises. θ is measured from vertical, l is the length of the pendulum, and g is the acceleration of gravity

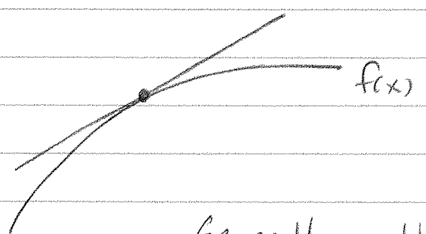
This is difficult to solve (ie. solve for θ) because it involves $\sin\theta$.

Physicists often use the approximation $\sin x \simeq x$ when x is small. Using this we have

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0$$

which is much easier to solve. In fact $\theta = \sin\sqrt{\frac{g}{l}}\,t$ and $\theta = \cos\sqrt{\frac{g}{l}}\,t$ both will solve this equation.

Why is the approximation $\sin x \simeq x$ legitimate?



If we construct the line tangent to a curve at some point, the line "approximates" the function in the neighborhood of the point.

Generally, the nearer we are to the point of tangency, the better the approximation.

Find the equation of the line tangent to $f(x)$ at $x = x_0$

- slope is $f'(x_0)$
- point on line is $(x_0, f(x_0))$

so $$y - f(x_0) = f'(x_0)(x - x_0)$$

or

$$y = f(x_0) + f'(x_0)(x - x_0)$$

The linear approximation (tangent line approximation or local linearization) of $f(x)$ at $x = x_0$ is

$$\boxed{L(x) = f(x_0) + f'(x_0)(x - x_0)}$$

Ex. Find the linear approximation of $\sin x$ near $x=0$.

$$f(x) = \sin x \quad \text{so} \quad f'(x) = \cos x$$

$$f'(0) = \cos 0 = 1$$
$$f(0) = \sin 0 = 0$$

so

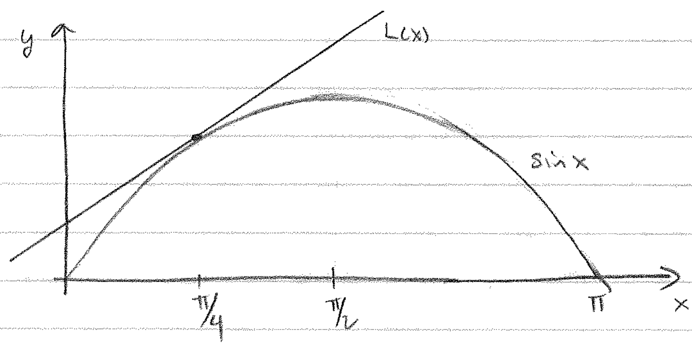
$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 0 + 1(x-0) \\ &= x \end{aligned}$$

This shows that $y = x$ does indeed approximate $y = \sin x$ near $x=0$.

Ex Repeat, but find approximation when $x = \pi/4$.

$$\begin{aligned} L(x) &= \sin \pi/4 + \cos \pi/4 \, (x - \pi/4) \\ &= \sqrt{2}/2 + \sqrt{2}/2 \, (x - \pi/4) \end{aligned}$$

$$L(x) = \frac{\sqrt{2}}{2}\left(x + 1 - \pi/4\right)$$



Ex. Find an approximation for $\sqrt{x}$ near $x=4$

$$f(x) = \sqrt{x} \quad \text{so} \quad f'(x) = \frac{1}{2\sqrt{x}}$$

$$\begin{aligned} L(x) &= \sqrt{4} + \frac{1}{2\sqrt{4}}(x-4) \\ &= 2 + \tfrac{1}{4}(x-4) \\ &= 2 + x/4 - 1 \\ &= x/4 + 1 \end{aligned}$$

$\sqrt{x} \approx \frac{x}{4} + 1$ when x is near 4.

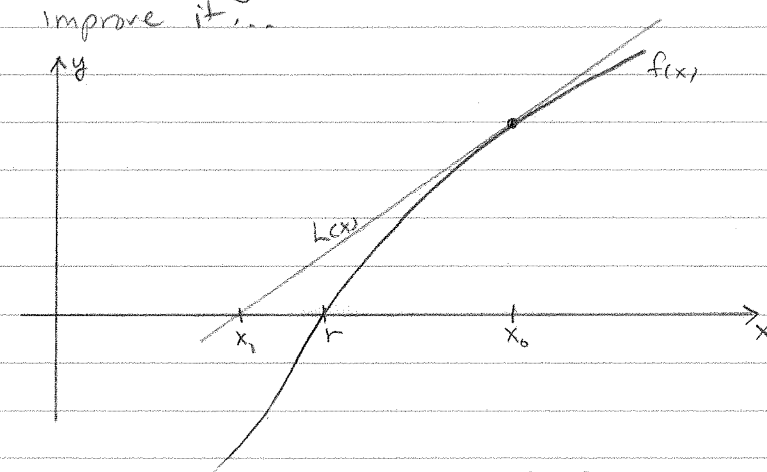
$\sqrt{4.1} \approx 2.0248$

$L(4.1) = 4.1/4 + 1 = 2.0250$

Now we turn our attention to the problem of finding roots, i.e., the values of x which satisfy $f(x)=0$.

We can often get an approximate value of x by examining the graph of $f(x)$. How could we get a more accurate (e.g. 8 decimal places) estimate?

We would like some way to take an initial estimate x_0 of the root r and improve it...



Idea: Use a linear approximation of $f(x)$ at x_0 to estimate the root.

Linear Approximations and Newton's Method

We can compute $L(x)$, the equation of the line tangent to $f(x)$ at $x = x_0$:

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

If we want to find the value of x where $L(x) = 0$ (since this is our "improved" estimate of the root, we can set $L(x_1) = 0$ and solve for x_1.

$$0 = L(x) = f(x_0) + f'(x_0)(x_1 - x_0)$$

$$f'(x_0)(x_1 - x_0) = -f(x_0)$$

$$x_1 - x_0 = -f(x_0)/f'(x_0)$$

$$x_1 = x_0 - f(x_0)/f'(x_0)$$

If we then repeat to find the line tangent to $f(x)$ at x_1 and use it to find an "even better" estimate x_2, we have started a process that should lead us to the root we seek.

Newton's Method

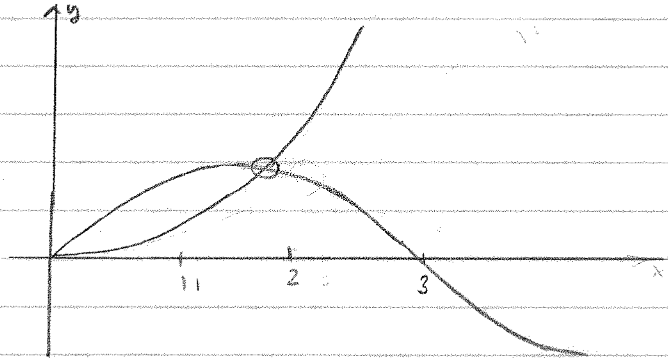
Given a differentiable function $f(x)$ and an initial estimate x_0 of a root of $f(x) = 0$, the iteration,

$$x_{n+1} = x_n - f(x_n)/f'(x_n) \qquad n = 0, 1, 2, \ldots$$

will converge to a root of $f(x) = 0$, if it converges at all.

Warning: the root found may not be the one sought.

Ex Solve $5\sin x = x^3$ for $x > 0$.



From the graph we expect a solution between $x=1$ and $x=2$.

To use Newton's Method, we need a function that is zero at the desired x value

$$f(x) = 5\sin x - x^3$$

does the trick.

$$f'(x) = 5\cos x - 3x^2$$

So our iteration is $x_{n+1} = x_n - \dfrac{5\sin x - x^3}{5\cos x - 3x^2}$

n	x_n
0	2.0
1	1.7547349
2	1.7067296
3	1.7048498
4	1.7048470
5	1.7048470

at this point our answer is not changing, we expect that we have found the root.

$$f(1.7048470) = -0.0000001459$$

So we've got very close to the answer!