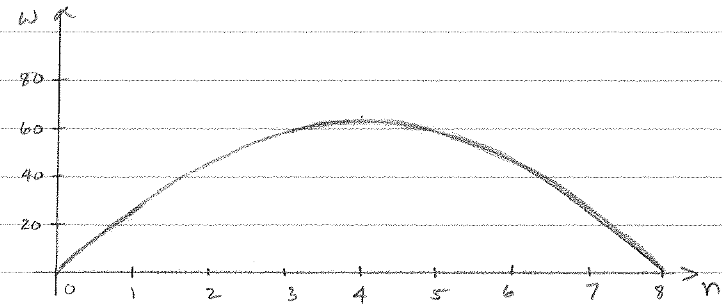
Maximum and Minimum Values

Suppose $W(n)$ gives the number of widgets per hour produced by a team of n people. Let $W(n) = 32n - 4n^2$. How many people should be on the team to maximize the number of widgets produced per hour?



From the graph it seems that the answer will be near $n=4$.

It looks like the maximum occurs where $W'(n) = 0$.

$$W'(n) = 32 - 8n$$

So

$$0 = 32 - 8n$$
$$8n = 32$$
$$n = 4.$$

So Widget production is maximized when 4 people are on the team.

Def. For a function f defined on a set S of real numbers an a number $c \in S$,

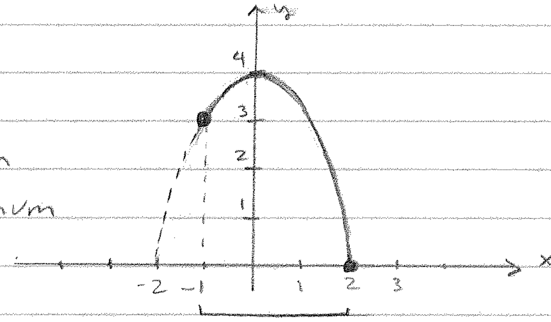
i) $f(c)$ is the <u>absolute maximum</u> of f on S if $f(c) \geq f(x)$ for all $x \in S$ and

ii) $f(c)$ is the <u>absolute minimum</u> of f on S if $f(c) \leq f(x)$ for all $x \in S$.

Maxima and minima are called <u>extrema</u> (singular is extremum).

Ex. Find all absolute extrema of $f(x) = 4 - x^2$ on $[-1, 2]$.

Graphing $f(x) = 4 - x^2$ we find



We see that the absolute maximum is $f(0) = 4$ and the absolute minimum is $f(2) = 0$.

Notice that one of these occured at a point where $f'(x) \neq 0$.

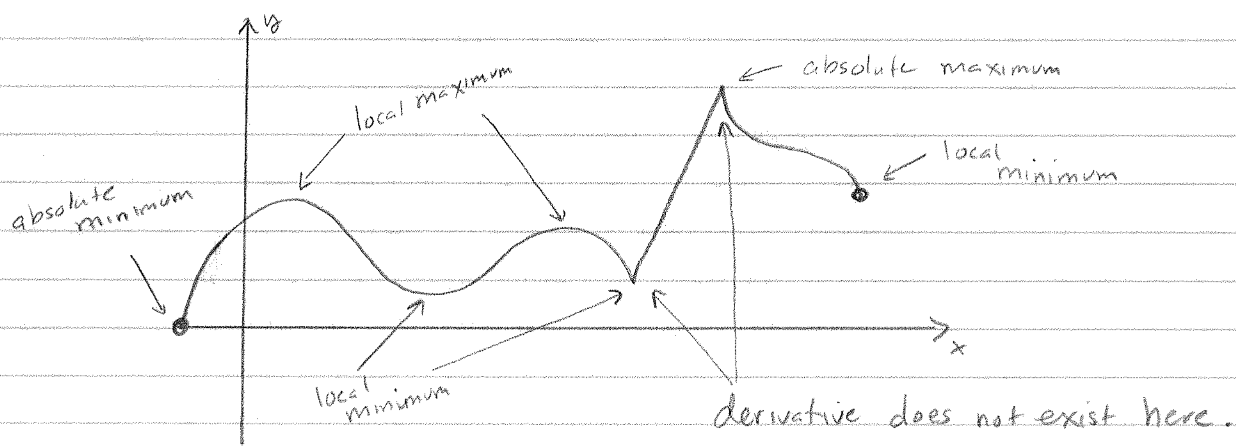
Thm: Extreme Value Theorem

A continuous function f defined on a closed, bounded interval $[a, b]$ attains both an absolute maximum and an absolute minimum on that interval.

Def. i) $f(c)$ is a local maximum of f if $f(c) \geq f(x)$ for all x in some open interval containing c.

ii) $f(c)$ is a local minimum of f if $f(c) \leq f(x)$ for all x in some open interval containing c.

Ex. Find all extrema of the function shown below



Maximum and Minimum Values

Given a continuous function f on a closed interval $[a, b]$, where can maxima and minima be found?

1. places where $f'(x) = 0$ (maybe...)
2. places where $f'(x)$ does not exist (maybe...)
3. endpoints a and b.

Def: A number c in the domain of f is called a <u>critical number or value</u> of if $f'(c) = 0$ or $f'(c)$ is undefined.

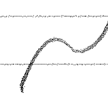
Find local extrema of the following functions

1. $f(x) = x^3 - 4x^2 - 3x + 7$

$$f'(x) = 3x^2 - 8x - 3 = 0$$
$$(3x + 1)(x - 3) = 0$$

so

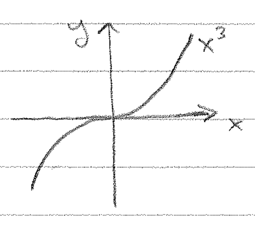
$x = -\frac{1}{3}$, $x = 3$ are the critical numbers
(no others since $f'(x)$ is always defined)

If we graph f we see it has the form  so

$$\boxed{\begin{array}{l} x = -\frac{1}{3} \rightarrow \text{local maximum} \\ x = 3 \rightarrow \text{local minimum.} \end{array}}$$

2. $f(x) = x^3$ $\qquad f'(x) = 3x^2 = 0 \rightarrow x = 0$

$x = 0$ is a critical number, but it is not a max or a min.



Find absolute extrema for $f(x) = x^2 e^{-x}$ on $[1,4]$

$$f'(x) = 2xe^{-x} + x^2 e^{-x}(-1)$$
$$= 2xe^{-x} - x^2 e^{-x}$$
$$= x(2-x)e^{-x}$$

This exists for all x in $[1,4]$.

$f'(x) = 0 \rightarrow \quad x(2-x)e^{-x} = 0 \rightarrow \quad x(2-x) = 0$

$x = 0, \; x = 2$ critical values

$x = 2$ is the only critical value in $[1,4]$

$$f(2) = 4e^{-2} = 4/e^2 \approx 0.54134$$

Check the endpoints

$$f(1) = 1e^{-1} = 1/e \approx 0.36788$$

$$f(4) = 16e^{-4} = 16/e^4 \approx 0.29305$$

Absolute maximum $(2, 4/e^2)$

Absolute minimum $(4, 16/e^4)$