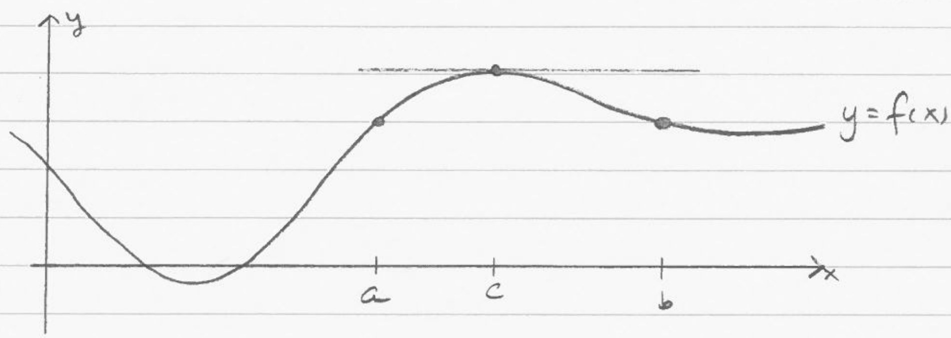
The Mean Value Theorem

We begin by stating and proving Rolle's Theorem

Rolle's Theorem

Suppose that f is continuous on the interval $[a,b]$, differentiable on the interval (a,b) and $f(a) = f(b)$. Then there is a number $c \in (a,b)$ such that

$$f'(c) = 0.$$



Proof

① If $f(x) = k$, a constant, then $f'(x) = 0$ for all $x \in (a,b)$ and so in this case the theorem is true.

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② If $f(x)$ is not constant but is continuous on $[a,b]$ then there must be at least one extremum (maximum or minimum) on (a,b).

"what goes up must come down"

If $f(x)$ increases away from $f(a)$ it must decrease before reaching $f(b)$.

If $f(x)$ decreases away from $f(a)$ it must increase before reaching $f(b)$.

Let $x = c$ be the location of this extremum. Since $f(x)$ is differentiable on (a,b) we know that $f'(c)$ exists.

$f'(c) \not< 0$ since this would mean f is decreasing at $x = c$ and so cannot be an extreme value

The Mean Value Theorem

Similarly, $f'(c) \ngtr 0$ since then f cannot be increasing at an extreme value.

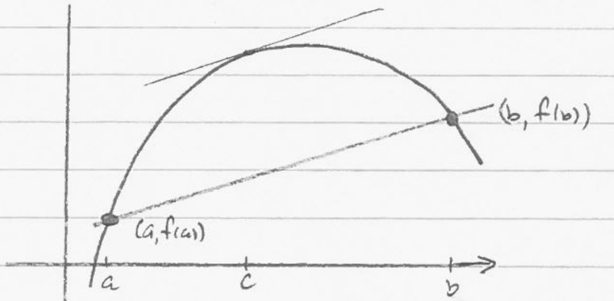
Therefore, if $f'(c)$ exists and $f'(c) \nless 0$ & $f'(c) \ngtr 0$ it must be that $f'(c) = 0$. ■

This leads us to one of the most important theorems in Calculus, the <u>Mean Value Theorem.</u>

> **Mean Value Theorem**
>
> Suppose that f is continuous on the interval $[a,b]$ and differentiable on the interval (a,b). Then there exists a number $c \in (a,b)$ such that
>
> $$f'(c) = \frac{f(b) - f(a)}{b-a}$$

In other words, this theorem says that there is a point on the interior of an interval I on which f is differentiable where the <u>instantaneous rate of change of f</u> equals the <u>average rate of change of f</u> over the entire interval

Proof



The slope of the line through $(a, f(a))$ and $(b, f(b))$ is

$$\frac{f(b) - f(a)}{b-a}$$

So the equation of this line is

$$y = \frac{f(b)-f(a)}{b-a}(x-a) + f(a).$$

Define $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a) - f(a).$

Note that $g(x)$ is both continuous on $[a,b]$ and differentiable on (a,b) because $f(x)$ and the polynomial $\frac{f(b)-f(a)}{b-a}(x-a) + f(a)$ both are.

Both $g(a)$ and $g(b)$ are zero:

$$g(a) = f(a) - \frac{f(b)-f(a)}{b-a}(a-a) - f(a) = f(a)-f(a) = 0$$

$$g(b) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) = f(b)-f(b)+f(a)-f(a) = 0$$

So we can use Rolle's Theorem to assert that there is a number $c \in (a,b)$ for which $g'(c) = 0$.

Computing $g'(x) = f'(x) - \frac{f(b)-f(a)}{b-a}$ we find

$$g'(c) = 0 = f'(c) - \frac{f(b)-f(a)}{b-a}$$

so

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$ ✓

Warning: Do not confuse the Intermediate Value Theorem with the Mean Value Theorem.

Two more theorems that follow from Rolle's Theorem and the Mean Value Theorem.

Thm: If f is continuous on the interval $[a,b]$, differentiable on the interval (a,b) and $f(x)=0$ has $n>0$ solutions in $[a,b]$, then $f'(x)=0$ has at least $(n-1)$ solutions in (a,b).

Proof: By Rolle's Theorem, between any two solutions of $f(x)=0$ there is a solution of $f'(x)=0$. Thus, given n solutions of $f(x)=0$, there must be at least $n-1$ solutions of $f'(x)=0$.

Ex Prove that $2x^4+x^2-6=0$ has exactly two solutions.

$$f(x) = 2x^4+x^2-6$$

$$f'(x) = 8x^3+2x = 2x(4x^2+1)$$

We can locate zeros of $f(x)$ by graphing or some simple numerical investigation

$$x=-2 \Rightarrow f(-2) = 2(-2)^4+(-2)^2-6 = 30 > 0$$

$$x=0 \Rightarrow f(0) = 0+0-6 = -6 < 0$$

$$x=2 \Rightarrow f(2) = 2(2)^4+(2)^2-6 = 30 > 0$$

So we find $2x^4+x^2-6=0$ has <u>at least two solutions</u>

Proof by contrapositive

Now consider $f'(x) = 2x(4x^2+1)$. The factor $4x^2+1$ is always positive so the <u>only zero</u> of $f'(x)$ is $x=0$.

If $f(x)$ had more than two solutions then, by our most recent theorem, $f'(x)$ would have more than one solution.

Therefore $2x^4+x^2-6=0$ has <u>no more than two solutions</u>

Since we have at least two solutions <u>and</u> no more than two solutions, we have exactly two solutions. ✓

Ex. Find the value of c to confirm the Mean Value Theorem for $f(x) = x^3+2x$ on $[0,1]$.

$a=0,\quad f(a) = 0^3+2\cdot 0 = 0$

$b=1,\quad f(b) = 1^3+2\cdot 1 = 3$

$$\frac{f(b)-f(a)}{b-a} = \frac{3-0}{1-0} = 3$$

Find c so $f'(c)=3$:

$$3c^2+2=3$$
$$3c^2=1$$
$$c=\pm\sqrt{1/3}$$

in $[0,1]$ we have $\boxed{c = 1/\sqrt{3}}$

The Mean Value Theorem

So what's the big deal? Why does the Mean Value Theorem warrant a special name?

Consider $f'(x) = 0$ for all x in some open interval. What can we conclude about f?

By the theorems in this section we know that $f(x)$ must be constant on I

Thm: Suppose that $f'(x) = 0$ for all x in some open interval I. Then, $f(x)$ is constant on I.

Corollary: Suppose that $g'(x) = f'(x)$ for all x in some open interval I. Then, for some constant c,

$$g(x) = f(x) + c \quad \text{for all } x \in I.$$

Proof Apply the theorem to $h(x) = g(x) - f(x)$

Perhaps most significant, the Mean Value Theorem is used to prove no less than the Fundamental Theorem of Calculus, as we shall see.

Ex #24. Use the MVT to show that $|\cos u - \cos v| \leq |u - v|$ for all u, v.

Let $f(x) = \cos x$ so $f'(x) = -\sin x$. Clearly f is continuous and differentiable so by the MVT we know there is a number c between u and v such that

$$\frac{f(u) - f(v)}{u - v} = f'(c)$$

Since $f'(x) = -\sin x$ we know $|f'(c)| \leq 1$ since $-1 \leq \sin x \leq 1$ for all x. Then

$$\left| \frac{f(u) - f(v)}{u - v} \right| \leq 1$$

so

$$|\cos u - \cos v| \leq |u - v| \quad \checkmark$$