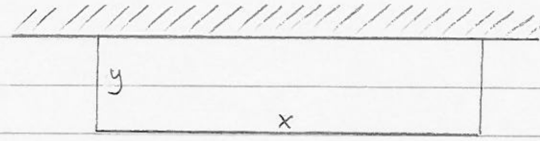
Optimization

Ex. Suppose you have 30 ft of fencing and want to fence in a rectangular garden next to a house. What should the dimensions of the rectangle be so that the garden is as large as possible?

(1) Draw a picture



(2) Label unknown quantities

(3) What do we want to maximize? Area

(4) What relationships can we develop between the variable in our diagram, the given information, and the quantity to be maximized?

$A = xy$ we need to get this in terms of a single variable.

$30 = y + x + y = x + 2y$

(5) Write quantity to be maximized in term of one variable

$x = 30 - 2y$

so

$A = (30-2y)y$

(6) Domain is $0 \le y \le 15$, endpoints $y=0$, $y=15$

(7) Find local extrema: $A' = -2y + 30 - 2y = 30 - 4y$

$A' = 0 \rightarrow 30 = 4y \quad y = 7.5$

$A(0) = 0, \quad A(7.5) = 112.5, \quad A(15) = 0$

(8) Absolute maximum is 112.5 sq ft. and occurs when $y = 7.5$ ft.

(9) Dimensions should be 7.5 ft × 15 ft.

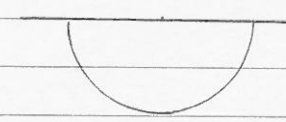
Note - see the guidelines at the beginning of section 3.7.

Note - always check to make sure you are answering the original question.

Note - pay special attention to endpoints.

Question: Could we enclose a larger garden with the same 30ft of fencing?

Use a semicircle.



$$P = 2\pi r \text{ so } P/2 = \pi r$$

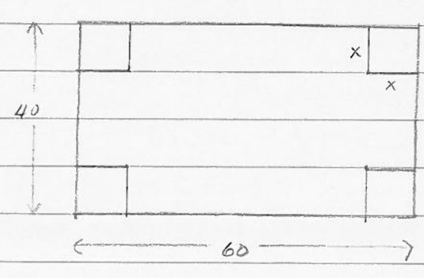
$$30 = \pi r \rightarrow r = 30/\pi$$

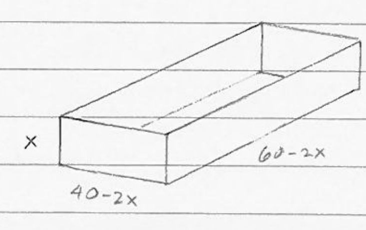
$$A = \tfrac{1}{2}\pi r^2 = \tfrac{1}{2}\pi\, \frac{30^2}{\pi^2} = \frac{900}{2\pi} \approx 143.24 \text{ sq. ft.}$$

Since $143.24 > 112.5$, we have a larger garden.

Ex. Find the largest open-topped box that can be constructed from a 40" x 60" piece of cardboard.

We assume here that "largest" means "largest volume".

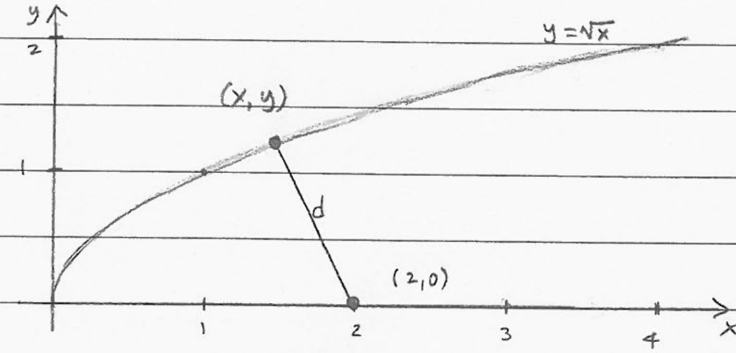




$$V = x\,(40-2x)(60-2x)$$

Clearly $x \geq 0$. We also see $x \leq 20$ so endpoints are $x=0$, $x=20$

Find the point on the curve $y = \sqrt{x}$ closest to the point $(2,0)$.



Use the distance formula between $(2,0)$ and the point (x,y) on the curve.

$$d = \sqrt{(x-2)^2+(y-0)^2} = \sqrt{(x-2)^2+(\sqrt{x}-0)^2}$$

We want to minimize d. Question: are d and d^2 minimized for the same value of x? Yes! This makes things easier: Find where $d^2 = (x-2)^2 + x$ is minimized.

Thm. If $f(x) > 0$ differentiable then the critical numbers of f and f^2 are the same.

Proof: $\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$

so $2f(x)f'(x) = 0$ gives $f'(x) = 0$ since $f(x) \neq 0$.

$$\frac{d}{dx} d^2 = 2(x-2) + 1 = 2x - 3 = 0 \Rightarrow x = \tfrac{3}{2}$$

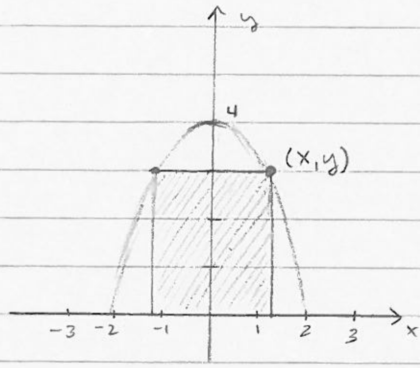
$$\frac{d^2}{dx^2} d^2 = 2 > 0$$ so sec. deriv. test says $x = \frac{3}{2}$ is location of local minimum.

When $x = \frac{3}{2}$, $y = \sqrt{\frac{3}{2}}$.

$\boxed{\text{Point on } y=\sqrt{x} \text{ closest to } (2,0) \text{ is } \left(\tfrac{3}{2}, \sqrt{\tfrac{3}{2}}\right)}$

Optimization

Ex Find the area of the largest rectangle that can be inscribed between the x-axis and the curve $y = 4 - x^2$.



We want to maximize $A = (2x)(y)$

$A = 2xy$

Need y in terms of x

$y = 4 - x^2$ so

$$A = 2x(4-x^2) = 8x - 2x^3$$

Domain is $0 \le x \le 2$.

$$A' = 8 - 6x^2 = 0 \Rightarrow 4 = 3x^2 \Rightarrow x^2 = 4/3 \Rightarrow x = \pm 2/\sqrt{3}$$

Only one of these x values is in the domain so we take $x = 2/\sqrt{3}$.

Check to make sure this is a local max

$A'' = -12x$ so $A'' < 0$ when $x = 2/\sqrt{3} \rightarrow$ local max

$A(0) = 0$, $A(2/\sqrt{3}) = \frac{4}{\sqrt{3}}(4 - \frac{4}{3}) = \frac{32}{3\sqrt{3}} \approx 6.1584$, $A(2) = 0$

Maximum area of inscribed rectangle is $\frac{32}{3\sqrt{3}}$ sq. units

Dimensions of rectangle are $\frac{4}{\sqrt{3}} \times \frac{8}{3}$

We want to maximize V, and V is already expressed in terms of a single variable.

$$\begin{aligned} V' &= (40-2x)(60-2x) + x(-2)(60-2x) + x(40-2x)(-2) \\ &= 2400 - 120x - 80x + 4x^2 - 120x + 4x^2 - 80x + 4x^2 \\ &= 12x^2 - 400x + 2400 \\ &= 4(3x^2 - 100x + 600) \end{aligned}$$

Solving

$$3x^2 - 100x + 600 = 0 \quad \text{gives} \quad x = \frac{50 \pm 10\sqrt{7}}{3}$$

So

$$x \simeq 25.49 \quad \text{or} \quad x \simeq 7.85$$

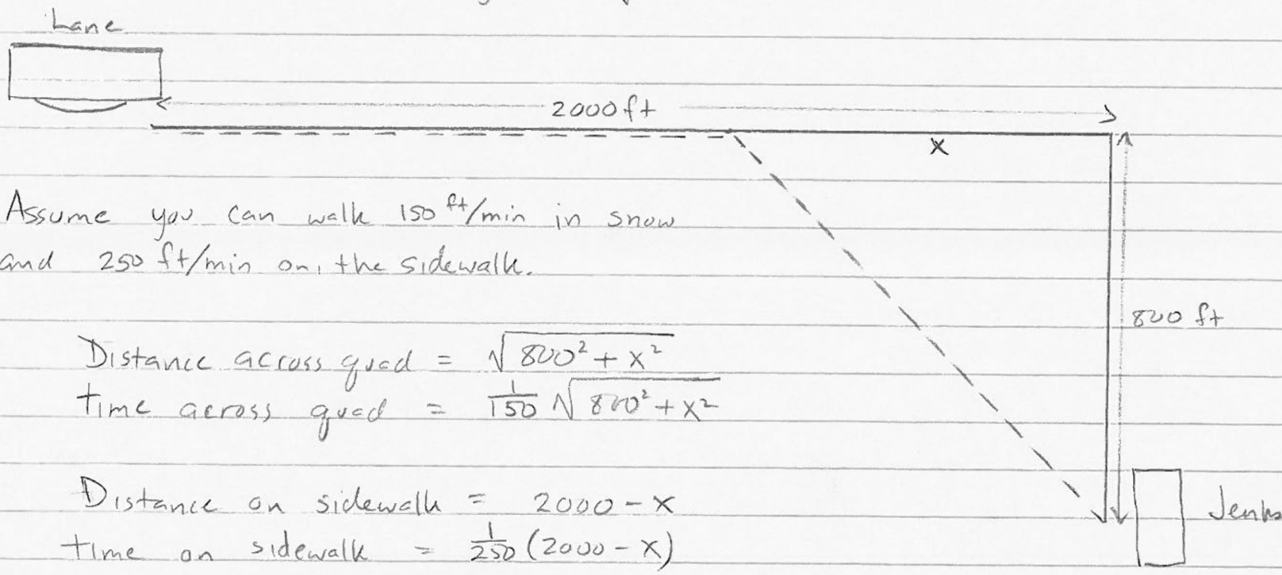
Only one of these values is in our domain $[0, 20]$.

$$V(0) = 0, \quad V\left(\tfrac{50-10\sqrt{7}}{3}\right) \simeq 8450.45 \text{ in}^3, \quad V(20) = 0.$$

The largest box is 8450.45 in^3 and has dimensions of approximately

$$7.85 \text{ in} \times 24.31 \text{ in} \times 44.31 \text{ in}$$

Ex. You're leaving Jenks on your way to Lane. There has been a recent snowfall so walking across the quad is much slower than using the plowed sidewalks. What course should you take to minimize the time of your trip?



Assume you can walk 150 ft/min in snow and 250 ft/min on the sidewalk.

Distance across quad $= \sqrt{800^2 + x^2}$

time across quad $= \frac{1}{150}\sqrt{800^2 + x^2}$

Distance on sidewalk $= 2000 - x$

time on sidewalk $= \frac{1}{250}(2000 - x)$

Total time $T = \frac{1}{150}\sqrt{800^2+x^2} + \frac{2000-x}{250}$ minutes

Clearly $0 \le x \le 2000$. Now find critical numbers for T.

$$T' = \frac{1}{150}\,\frac{1}{2}\left(800^2+x^2\right)^{-1/2}(2x) - \frac{1}{250} = 0$$

$$= \frac{x}{150\sqrt{800^2+x^2}} - \frac{1}{250} = 0$$

$$\frac{x}{\sqrt{800^2+x^2}} = \frac{150}{250} = \frac{3}{5}$$

$$\frac{x^2}{800^2+x^2} = \frac{9}{25} \rightarrow \quad 25x^2 = 9\cdot 800^2 + 9x^2$$

$$16x^2 = 9\cdot 800^2$$
$$4x = 3\cdot 800$$
$$x = 600 \text{ ft.}$$

$T(0) = 13.\overline{3}$ min, $T(600) = 12.2\overline{6}$ min, $T(2000) \simeq 14.36$ min

Best time is obtained when $x = 600$ft. This means you should aim for a point in front of the entrance to the new science center.