Polynomials & Rational Functions

Topics discussed in text not dealt with below

- integers ($\mathbb{Z}$), rational numbers ($\mathbb{Q}$),
- irrational numbers, real numbers ($\mathbb{R}$)

- open & closed intervals
- inequalities
- absolute value, triangle inequality
- distance formula
- slope
- point-slope form: $y - y_0 = m(x - x_0)$ or $y = m(x-x_0) + y_0$
- slope-intercept form: $y = mx + b$
- parallel lines (same slope)
- perpendicular lines (negative reciprocal slope)

Def A <u>function</u> f is a rule that assigns exactly one element y in a set B to each element x in a set A.

We write $y = f(x)$.

A is the <u>domain</u> of f and B is the <u>range</u> of f.

Note: if the domain is not specified, we assume it is the largest set of real numbers for which the function is defined.

We call x the <u>independent variable</u> and call y the <u>dependent variable</u>.

The <u>graph</u> of $f(x)$ is the set of points (x, y) that satisfy $y = f(x)$.

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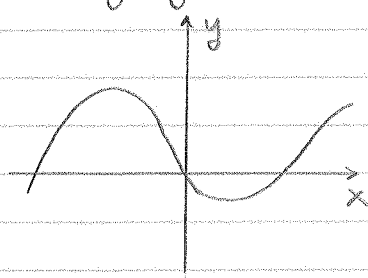
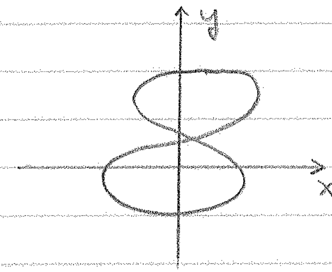
Not every graph is a function



This graph does represent a function



This graph does not represent a function

You can use the vertical line test to determine if a function represents a graph

if a vertical line can be drawn that touches the graph at more than one point the the graph does not describle a function.

Def. A polynomial is any function that can be written in the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

Where $a_0, a_1, \dots, a_n$ are real numbers (coefficients) with $a_n \neq 0$ and $n \geq 0$ is an integer (degree).

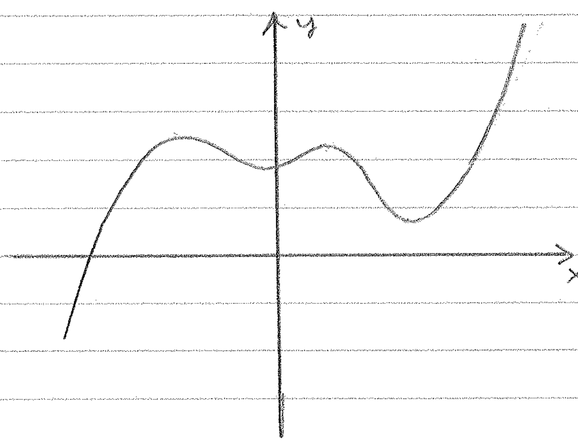
Polynomials	Not Polynomials
$3x + 1$	$1/x$
$(x+2)^3 = x^3 + 6x^2 + 12x + 8$	$(x+2)^{1/2} = \sqrt{x+2}$
$1 + x + 5x^2 + 3x^{15}$	$4x^{-2} + 2x$
5	
$\frac{1}{2}x^2 + \frac{3}{5}$	

The degree of a polynomial is often used to classify the polynomial.

Polynomials & Rational Functions

degree	name
0	constant
1	linear
2	quadratic (why "quad"?)
3	cubic
4	quartic
5	quintic

It is often possible to determine the degree of a polynomial from its graph. It is always possible to determine a lower bound on the degree.



This will have odd degree and is at least degree 5.

Goes to $+\infty$ one one side and $-\infty$ on the other

Degree is at least one more than the number of "humps".

Def Any function that can be written in the form

$$f(x) = p(x)/q(x)$$

where p and q are polynomials is called a rational function.

rational as in "ratio".

Since p and q are polynomials, which are defined for all x, p/q is defined everywhere that $q(x) \neq 0$.

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Ex. What is the domain of

$$f(x) = \frac{x^2+1}{x^2-1} \ ?$$

Here $q(x) = x^2-1$ which is zero when $x=1$ and $x=-1$.

Therefore the domain is

$$\{x \in \mathbb{R} \mid x \neq 1 \text{ and } x \neq -1\}$$

or

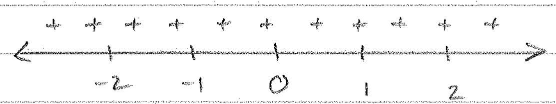
$$(-\infty,-1) \cup (-1,1) \cup (1,\infty)$$

or

All reals except 1 and -1.

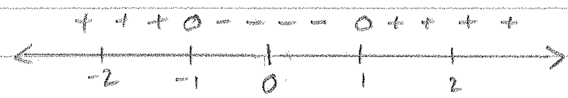
Note also that we can learn something about the graph of this function before we graph it

$p(x) = x^2+1$



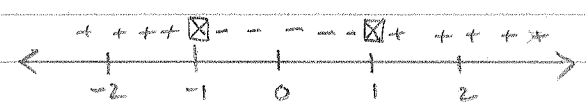
always positive

$q(x) = x^2-1$

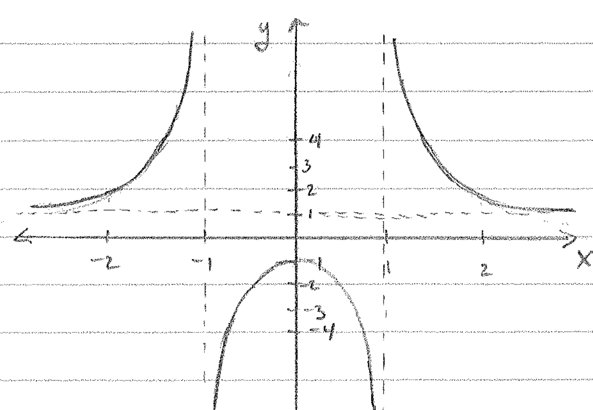


so

$f(x) = x^2+1 / x^2-1$



where ⊠ means not defined



notice the scale on the two axes is not the same

We often will need to find the zeros or roots of a function, i.e. find x so $f(x)=0$.

When $f(x)$ is a polynomial, rational function, or some simple function based on these we may be able to find the roots by factoring.

Example: Find the zeros of $f(x)=\sqrt{\dfrac{2x^2-x-3}{x^2-4}}$

① Note that we can ignore the radical since $f(x)$ will be zero exactly where the argument of the square root is zero

② $\dfrac{2x^2-x-3}{x^2-4}$ has zeros where the numerator $2x^2-x-3$ has zeros - as long as denominator is not 0

③ $2x^2-x-3$: $(2)(-3)=-6$
→ find a factorization of -6 that sums to -1 (coefficient of x)

$$-6=(-1)(6)=(-2)(3)=(2)(-3)=(1)(-6)$$

$$2+(-3)=-1 \quad \text{so}$$

$$\begin{aligned}2x^2-x-3 &= 2x^2+2x-3x-3\\ &= 2x(x+1)-3(x+1)\\ &=(2x-3)(x+1)\end{aligned}$$

Since $(2x-3)(x+1)=0$ only when $2x-3$ or $x+1=0$ we find roots are $x=3/2$ and $x=-1$.

$\therefore$ $\boxed{\text{zeros of } f(x)=\sqrt{\dfrac{2x^2-x-3}{x^2-4}} \text{ are } x=3/2,\ x=-1}$