The Power Rule

Suppose $f(x) = c$.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

$$\boxed{\text{For any constant } c, \quad \frac{d}{dx} c = 0}$$

Suppose $f(x) = x$

$$f'(x) = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

$$\boxed{\frac{d}{dx} x = 1}$$

Suppose $f(x) = x^n$ for integer $n > 0$

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$$

How shall we proceed? We can use the Binomial Theorem.

For any real numbers a and b and positive integer n

$$(a+b)^n = \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2 + \cdots + \binom{n}{n-1} a^1 b^{n-1} + \binom{n}{n} a^0 b^n$$

where $\binom{n}{k}$ is the binomial coefficient

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

and can also be found from Pascal's Triangle.

We have, then

$$(x+h)^n = x^n + n x^{n-1} h + \frac{n(n-1)}{2} x^{n-2} h^2 + \cdots + n x h^{n-1} + h^n$$

So

$$(x+h)^n - x^n = nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n$$

and we note the first term has h, the second term has h^2, etc., on up to the last term with h^n.

Inserting back into our limit we find

$$f'(x) = \lim_{h\to 0} \frac{h\left(nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right)}{h}$$

$$= \lim_{h\to 0}\left(nx^{n-1} + \underbrace{\frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}}_{\text{all these terms have } h \text{ as a factor so they go to zero as } h\to 0}\right)$$

$$\therefore\ f'(x) = nx^{n-1}$$

$$\boxed{\frac{d}{dx}x^n = nx^{n-1} \quad \text{for integer } n>0}$$

The Power Rule

Suppose $f(x) = c\,g(x)$ where c is a constant

$$f'(x) = \lim_{h\to 0}\frac{c\,g(x+h) - c\,g(x)}{h}$$

$$= \lim_{h\to 0} c\,\frac{g(x+h)-g(x)}{h}$$

$$= c\lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = c\,g'(x)$$

$$\boxed{\frac{d}{dx}c\,f(x) = c\frac{df}{dx} \quad \text{for any constant } c}$$

Ex $\frac{d}{dx}(3x^2) = 3\frac{d}{dx}x^2 = 3\cdot 2x^1 = \underline{\underline{6x}}$

Ex $\frac{d}{dx}(2.5x^6) = 2.5\cdot 6x^5 = \underline{\underline{15x^5}}$

The Power Rule

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \lim_{h\to 0} \frac{[f(x+h) \pm g(x+h)] - [f(x) \pm g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)] \pm [g(x+h) - g(x)]}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h\to 0} \frac{g(x+h) - g(x)}{h}$$

$$= f'(x) \pm g'(x)$$

$$\boxed{\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)}$$

Ex $\frac{d}{dx}(2x^5 + 7x^2 + 2) = 2\cdot 5x^4 + 7\cdot 2x^1 + 0$

$= \underline{\underline{10x^4 + 14x}}$

Ex $\frac{d}{dx}\frac{3}{x} = 3\frac{d}{dx}\frac{1}{x} = 3\left(-\frac{1}{x^2}\right) = \underline{\underline{-3/x^2}}$

Ex Given $\frac{d}{dx}\sqrt{x} = 1/2\sqrt{x}$, find $\frac{d}{dx}(3x^2 + 5\sqrt{x})$

$\frac{d}{dx}(3x^2 + 5\sqrt{x}) = 6x + 5/2\sqrt{x} = \underline{\underline{6x + \frac{5}{2\sqrt{x}}}}$

Ex Suppose the position of a particle is given by

$s(t) = 2t^2 + 3t + 10$ meters

where t is measured in seconds.

$s'(t) = v(t) = 4t + 3$ is velocity and has units of meters per second (think $ds/dt \to$ meters/sec)

If we differentiate again we have

$s''(t) = 4 \; m/s^2$ which is acceleration

Here we've found a second derivative, denoted $s''(t)$.

The Power Rule

	Order of differentiation	Newtonian Notation	Leibniz Notation
$y = f(x)$	0	$f(x)$	y
	1	$f'(x)$	dy/dx
	2	$f''(x)$	d^2y/dx^2
	3	$f'''(x)$	d^3y/dx^3

Ex. Find $f'''(x)$ if $f(x) = x^2 + 2x + 7$

$$f'(x) = 2x + 2$$
$$f''(x) = 2$$
$$f'''(x) = 0$$