The Product and Quotient Rules

Suppose we want to compute $\frac{d}{dx} f(x)g(x)$, i.e. the derivative of the product of f and g.

One way to proceed is to expand $f(x)g(x)$ into a single function. This is possible for many functions like polynomials or some functions involving square roots.

We will soon be working with other functions (e.g. $\ln x$, e^x, $\sin x$) that we cannot do this with.

Let use use the definition of the derivative to see how to compute $[fg]'$.

$$\frac{d}{dx} f(x)g(x) = \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

add and subtract same quantity

$$= \lim_{h\to 0} \frac{f(x+h)g(x+h) \overbrace{- f(x)g(x+h) + f(x)g(x+h)} - f(x)g(x)}{h}$$

$$= \lim_{h\to 0} \frac{[f(x+h) - f(x)]g(x+h) + [g(x+h) - g(x)]f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} g(x+h) + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} f(x)$$

$$= \left[\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}\right]\left[\lim_{h\to 0} g(x+h)\right] + \left[\lim_{h\to 0} \frac{g(x+h) - g(x)}{h}\right]\left[\lim_{h\to 0} f(x)\right]$$

$$= f'(x)g(x) + g'(x)f(x)$$

Thm. (Product Rule)

Suppose f and g are differentiable at x. Then

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x)$$

Ex $\frac{d}{dx} x^2(2x^3+5x) = 2x(2x^3+5x) + x^2(6x^2+5) = 4x^4+10x^2+6x^4+5x^2$

$$= \boxed{10x^4 + 15x^2}$$

check. $\frac{d}{dx} 2x^5 + 5x^3 = 10x^4 + 15x^2$ ✓

Ex $\frac{d}{dx}\, 3\sqrt{x}\,(5x^2+2)$

$$= \frac{3}{2\sqrt{x}}(5x^2+2) + 3\sqrt{x}\,(10x) = \frac{3(5x^2+2)}{2\sqrt{x}} + 30\sqrt{x}\cdot x$$

$$= \frac{15x^2+6}{2\sqrt{x}} + 30x\sqrt{x} = \frac{15x^2+60x^2+6}{2\sqrt{x}} = \boxed{\frac{75x^2+6}{2\sqrt{x}}}$$

Please do "reasonable" simplifications after using the product rule as in these examples.

Now, how can we compute $\frac{d}{dx} f(x)/g(x)$? Using the definition

$$\frac{d}{dx}\left[f(x)/g(x)\right] = \lim_{h\to 0} \frac{f(x+h)/g(x+h) - f(x)/g(x)}{h}$$

$$= \lim_{h\to 0} \frac{\dfrac{f(x+h)\,g(x) - f(x)\,g(x+h)}{g(x+h)\,g(x)}}{h}$$

$$= \lim_{h\to 0} \frac{f(x+h)\,g(x) \overbrace{- f(x)\,g(x) + f(x)\,g(x)}^{\text{add \& subtract}} - f(x)\,g(x+h)}{h\, g(x+h)\, g(x)}$$

$$= \lim_{h\to 0} \frac{\dfrac{f(x+h)-f(x)}{h}\,g(x) - \dfrac{g(x+h)-g(x)}{h}\,f(x)}{g(x+h)\,g(x)}$$

$$= \frac{g(x)\left[\lim_{h\to 0}\dfrac{f(x+h)-f(x)}{h}\right] - f(x)\left[\lim_{h\to 0}\dfrac{g(x+h)-g(x)}{h}\right]}{\lim_{h\to 0} g(x+h)\,g(x)}$$

$$= \frac{g(x)\,f'(x) - f(x)\,g'(x)}{[g(x)]^2}$$

provided $g(x) \neq 0$

Thm (Quotient Rule)

Suppose that f and g are differentiable at x and $g(x) \neq 0$. Then

$$\frac{d}{dx}\frac{f(x)}{g(x)} = \frac{f'(x)\,g(x) - f(x)\,g'(x)}{[g(x)]^2}$$

Ex Evaluate $\dfrac{d}{dx}\dfrac{2x^2+1}{1-3x} = \dfrac{(4x)(1-3x) - (2x^2+1)(-3)}{(1-3x)^2}$

$$= \frac{4x - 12x^2 + 6x^2 + 3}{(1-3x)^2} = \boxed{\frac{-6x^2+4x+3}{(1-3x)^2}}$$

Ex. Prove the power rule if $n<0$.

Let $m=-n$ so $m>0$ and $x^n = 1/x^m$

$$\frac{d}{dx}x^n = \frac{d}{dx}\left[1/x^m\right] = \frac{x^m\cdot(0) - 1\cdot m x^{m-1}}{[x^m]^2}$$

$$= -m\,\frac{x^{m-1}}{x^{2m}}$$

$$= -m\,x^{m-1-2m}$$

$$= -m\,x^{-m-1}$$

$$= n\,x^{n-1} \checkmark$$

Things to Remember

1. Product rule has a "+"; order of functions is not important
2. Quotient rule has a "−"; order of functions <u>is</u> important!

Come up with some mechanism to remember the order;
"Lo-de-hi minus hi-de-lo over lo squared"