Rates of Change in Economics and the Sciences

Consider the cost of producing x items.

$$C(x) = \underbrace{\text{fixed cost}}_{\text{independent of } x} + \underbrace{\text{variable cost}}_{\text{depends on } x}$$

Economists define the Marginal Cost to be the derivative of the cost function with respect to the quantity produced.

$$\text{Marginal Cost} = dC/dx$$

Ex. Suppose you want to make buttons for your favorite cause (I LOVE MATH!). A button machine can be bought for \$200 and button materials are 22¢ per button. What is the cost function $C(x)$? How much does it cost to make 750 buttons?

$$C(x) = 200 + 0.22x$$

$$C(750) = 200 + 165 = \$365 \text{ to make 750 buttons}$$

Notice that $C'(x) = 0.22$ dollars/button is the cost of producing a single button.

$$\boxed{\text{Marginal Cost is a measure of the cost of producing one additional item}}$$

Notice that while Cost is a function on the integers and is NOT really a continuous function, we are "extending" the definition of $C(x)$ to be a continuous real valued function on the real numbers.

This works well in many cases and allows us to use Calculus.

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In many real situations the rate at which the cost increases is ~~not~~ constant but actually increases as the number of items produced increases. Examples include machine maintence, employee raises, etc.

Lets take $C(x) = 200 + 0.22x + 0.0004x^2$

The marginal cost is $C'(x) = 0.22 + 0.0008x$

The marginal cost at $x = 750$ is $C'(750) = 0.22 + 0.6 = \$0.82$

The cost of producing the 750^{th} item is $C(750) - C(749) = 590 - 589.18$
$= \$0.8196$

While the cost of producing the 751^{st} item is $C(751) - C(750)$
$= 590.8204 - 590$
$= \$0.8204$

We see that the marginal cost is a very good estimator of the cost of producing a single object.

We define the average cost of producing x objects as

$$\overline{C}(x) = C(x)/x$$

The average cost of producing 750 buttons is $C(750)/750 = \frac{590}{750} = 0.78\overline{6}$ or $78\frac{2}{3}$ cents per button.

Notice the units on average cost are dollars per unit.

Q: How many buttons should we produce to minimize the average cost?

Want to minimize $\overline{C}(x)$.

$$\overline{C}'(x) = \frac{x \cdot C'(x) - C(x)}{x^2} = \frac{C'(x)}{x} - \frac{C(x)}{x^2}$$

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In the case of our button business, we find

$$\bar{C}'(x) = \frac{0.22 + 0.0008x}{x} - \frac{200 + 0.22x + 0.0004x}{x^2}$$

$$= \frac{-200 + 0.0004x^2}{x^2} = 0.0004 - \frac{200}{x^2}$$

To find the critical value

$$\bar{C}'(x) = 0 \Rightarrow \quad 0.0004 - \frac{200}{x^2} = 0$$

$$x^2 = \frac{200}{0.0004} = 500{,}000$$

$$x = \sqrt{500{,}000} \simeq 707.1$$

Since $\bar{C}''(x) = \frac{400}{x^3} > 0$ when $x > 0$, we know the critical value corresponds to a local minimum. Thus, the average cost per button is minimized when 707 buttons are produced.

Elasticity of Demand

Suppose the demand (# items consumed) is x and that this is a function of price p:

$$x = f(p)$$

Often as p increases x decreases and viceversa.

The relative change in price is computed $\frac{\Delta p}{p}$ when the price changes by some amount Δp.

The relative change in demand is computed $\frac{\Delta x}{x}$.

The elasticity of demand at price p is defined as $\frac{\Delta x}{x} \Big/ \frac{\Delta p}{p}$ for small values of Δp.

Using calculus we define this as E:

$$E = \lim_{\Delta p \to 0} \frac{\frac{\Delta x}{x}}{\frac{\Delta p}{p}}$$

Notice that $\Delta x = f(p + \Delta p) - f(p)$

inelastic price
- demand not affected strongly by price

elastic price
- demand strongly affected by price.

so

$$E = \lim_{\Delta p \to 0} \frac{\frac{f(p+\Delta p) - f(p)}{x}}{\frac{\Delta p}{p}} = \frac{p}{x} \lim_{\Delta p \to 0} \frac{f(p+\Delta p) - f(p)}{\Delta p} = \frac{p}{f(p)} f'(p)$$

$$\boxed{E = \frac{p}{f(p)} f'(p)}$$

Ex Suppose we know $f(p) = 5400e^{-0.8p}$ for a certain type of button.

What is the elasticity of demand when the price is \$1? when it is \$2?

$$E = \frac{p}{5400e^{-0.8p}} \cdot 5400e^{-0.8p}(-0.8)$$

$$= -0.8p$$

$-1 < E < 0$	Demand is relatively inelastic
$E < -1$	Demand is relatively elastic

When $p = 1$, $E = -0.8$

When $p = 2$, $E = -1.6$

Revenue is computed (price)×(quantity sold) or $p \cdot x = p f(p)$.

The revenue when $p = 1$ is $1 \cdot 5400e^{-0.8} \approx \$2,426.38$

" " " $p = 2$ is $2 \cdot 5400e^{-0.8 \cdot 2} \approx \$2,180.48$

A natural question is, what price will maximize revenue?

$$R = pf(p) \quad \text{so} \quad \frac{dR}{dp} = f(p) + pf'(p)$$

In our example $R' = 5400e^{-0.8p} + p \cdot 5400e^{-0.8p} \cdot (-0.8)$

$$= 5400e^{-0.8p}(1 - 0.8p)$$

Setting $R' = 0$ gives $0 = 5400e^{-0.8p}(1 - 0.8p)$

$$= 1 - 0.8p$$

$p = 1.25$ $\quad R(1.25) \approx \$2,483.19$

Therefore, the revenue is maximized when the price per button is \$1.25