Sums and Sigma Notation

Consider $1+2+3+4+5+6+7+8+9+10$, the sum of the first ten positive integers.

Since expressing long sums is frequently done, mathematicians have developed a shorthand way to do this. The sum above can be written

$$\sum_{i=1}^{10} i$$

This is called summation notation. The variable i is the index of summation, and each term in the sum is generated by allowing it to take on each value from the starting value to the ending value.

Ex $\displaystyle\sum_{i=0}^{5} 2i = 0+2+4+6+8+10$

Ex $\displaystyle\sum_{i=1}^{5} i^2 = 1^2+2^2+3^2+4^2+5^2 = 1+4+9+16+25$

Using summation notation does not make evaluating the sum any easier — it merely makes it more convenient to write it out.

Ex Write the sum of the first 5 positive odd integers using summation notation

$$1+3+5+7+9 = \sum_{i=1}^{5}(2i-1)$$

We found a formula for the general term, $2i-1$, then selected endpoints.

Ex Reexpress the last sum using $2i+1$ as the general term

$$\sum_{i=0}^{4} 2i+1$$

Sums and Sigma Notation

Ex Evaluate $\sum_{i=2}^{4} i/3$

$$= 2/3 + 3/3 + 4/3 = 9/3 = 3$$

Ex Evaluate $\sum_{i=1}^{100} 10 = \underbrace{10 + 10 + \cdots + 10}_{100 \text{ terms}} = 100(10) = 1000.$

Ex Evaluate $\sum_{i=1}^{100} i = 1 + 2 + 3 + \cdots + 99 + 100$

When given this task, a 10 year old Carl Gauss noticed

$$1 + 2 + 3 + 4 + 5 + \cdots + 96 + 97 + 98 + 99 + 100$$

could be "folded to get

$$\begin{array}{l} 1 + 2 + 3 + \cdots + 49 + 50 + \\ 100 + 99 + 98 + \cdots + 52 + 51 \end{array}$$

adding vertically, we get

$$\underbrace{101 + 101 + 101 + \cdots + 101 + 101}_{50 \text{ terms}}$$

So the sum is $50(101) = 5050$.

In general, we can compute $\sum_{i=1}^{n} i$ in a similar way.

$$\begin{array}{rl} \sum_{i=1}^{n} i = & 1 + 2 + 3 + \cdots + (n-1) + n \\ + \quad \sum_{i=1}^{n} i = & n + (n-1) + \cdots + 2 + 1 \\ \hline \end{array}$$

$$2\sum_{i=1}^{n} i = \underbrace{(n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1)}_{n \text{ terms}}$$

$$= n(n+1)$$

$$\therefore \boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

Sums and Sigma Notation

Often it is necessary to write a sum without writing the general term, either because it is complicated or unknown.

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \cdots + a_n$$

where a_i is the general term. For example a_i might be i or $2i+1$ or i^2.

Thm For any constants c and d.

$$\sum_{i=1}^{n} (ca_i + db_i) = c\sum_{i=1}^{n} a_i + d\sum_{i=1}^{n} b_i$$

This is easy to prove and follows directly from the rules of addition.

Ex $$\sum_{i=1}^{500} 2i-1 = 2\sum_{i=1}^{500} i - \sum_{i=1}^{500} 1$$

$$= 2\,\frac{500 \cdot 501}{2} - 500 \cdot 1$$

$$= 250500 - 500 = 250{,}000$$

We conclude with one more useful formula

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

which we can prove using <u>mathematical induction</u>.

The proof is in two parts

① Show the <u>base case</u>, in this case when $n=1$

② Show that if the formula is correct for $n=k$ then it must be correct for $n=k+1$

① Base Case : $n=1$

$$\sum_{i=1}^{1} i^2 = \underline{1} \qquad \frac{1\cdot 2\cdot 3}{6} = \frac{6}{6} = \underline{1}$$

formula works when $n=1$

② Assume $\displaystyle\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$

and show $\displaystyle\sum_{i=1}^{k+1} i^2 = \frac{(k+1)(k+2)(2k+3)}{6}$

$$\sum_{i=1}^{k} i^2 = \frac{k(k+1)(2k+1)}{6}$$

$$\sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$\sum_{i=1}^{k+1} i^2 = \left[\frac{k(2k+1)}{6} + (k+1)\right](k+1)$$

$$= \frac{k(2k+1)+6(k+1)}{6}(k+1)$$

$$= \frac{2k^2+7k+6}{6}(k+1)$$

$$= \frac{(k+2)(2k+3)}{6}(k+1)$$

This step is easy since we know what factorization we should look for.

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

✓

Ex $\displaystyle\sum_{i=0}^{20} i^2+3i+5 = 0^2+3\cdot 0+5 + \sum_{i=1}^{20} i^2+3i+5$

$$= 5 + \sum_{i=1}^{20} i^2 + 3\sum_{i=1}^{20} i + \sum_{i=1}^{20} 5$$

$$= 5 + \frac{(20)(21)(41)}{6} + 3\,\frac{20\cdot 21}{2} + 20\cdot 5$$

$$= 5 + 10\cdot 7\cdot 41 + 3\cdot 10\cdot 21 + 100$$

$$= 5 + 2870 + 630 + 100 = 3{,}605$$

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Sums and Sigma Notation

Ex Compute the sum $\sum_{i=1}^{n} f(x_i)\,\Delta x$ if

$$f(x) = 1+x^2, \quad x = 0.0, 0.5, 1.0, 1.5, 2.0, \quad \Delta x = 0.5, \quad n=5$$

$$\sum_{i=1}^{5} f(x_i)\,\Delta x = (1+0.0^2)(0.5) + (1+0.5^2)(0.5) + (1+1.0^2)(0.5) + (1+1.5^2)(0.5) + (1+2.0^2)(0.5)$$

$$= (1.00)(0.5) + (1.25)(0.5) + (2.00)(0.5) + (3.25)(0.5) + (4.00)(0.5)$$

$$= 6.25$$

Note x_i is given by $x_1 = 0.0$, $x_2 = 0.5$, $x_3 = 1.0$, $x_4 = 1.5$, $x_5 = 2.0$