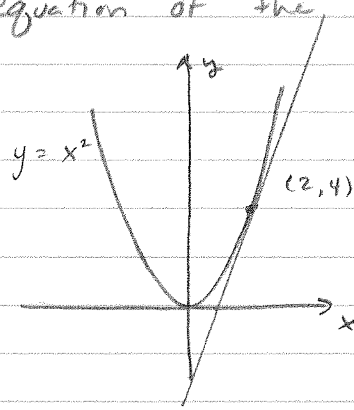
Tangent Lines and Velocity

Consider $y = x^2$. How can we find the equation of the line tangent to this curve at the point $(2,4)$?



Since we know $(2,4)$ is on the line, we only need to know the slope m and then we can use the point-slope equation

$$y - y_1 = m(x - x_1)$$

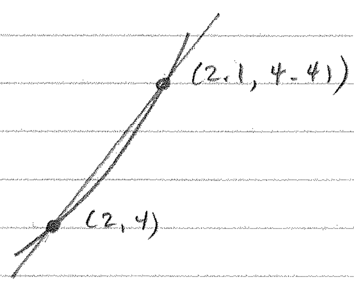
to find the equation of the line.

So - how can we find the slope?

1. Estimate from graph - looks like slope is 4
2. Estimate numerically

pick a point near $(2,4)$

Say $x = 2.1$. Then $y = x^2 = 2.1^2 = 4.41$



The line through the curve at two points is called a <u>secant line</u>. The slope of this secant is computed as

$$\frac{\Delta y}{\Delta x} = \frac{4.41 - 4}{2.1 - 2} = \frac{0.41}{0.1} = \boxed{4.1}$$

which is near 4.

How could we improve our estimate of the tangent's slope?

Use a point closer to (2.4). Try $(2.01, 4.0401)$

$$\frac{\Delta y}{\Delta x} = \frac{4.0401 - 4}{2.01 - 2} = \frac{0.0401}{0.01} = \boxed{4.01}$$

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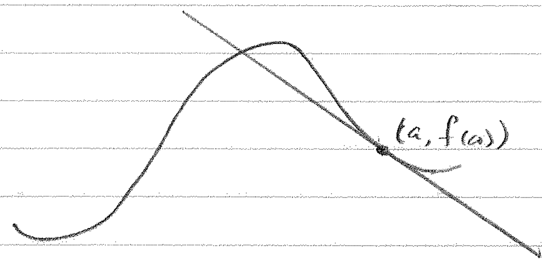
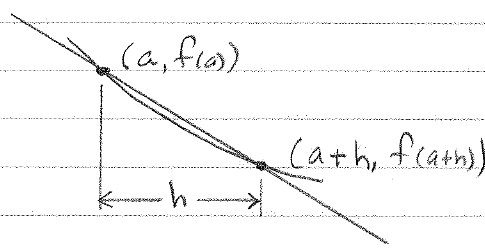
At this point, it looks like the slope of the tangent line is 4. How can we prove it?

<u>Limits!</u>

Suppose we want to find the slope of a line tangent to the graph of $y=f(x)$ at $(a, f(a))$.



We can pick a point on the curve near $x=a$, say $x=a+h$, and find the slope of the secant through these two points



$$m_{sec} = \frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{a+h-a} = \frac{f(a+h)-f(a)}{h}$$

This computation is the same one we used in our last example.

It appears that m_{sec} is close the slope we want, and...

<u>it will get closer as $h \to 0$.</u>

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The expression $\frac{f(a+h)-f(a)}{h}$ is known as the <u>difference quotient of $f(x)$ at $x=a$</u>.

If we really want m_{tan}, the slope of the tangent line, we can take the limit of the difference quotient as $h \to 0$:

$$m_{tan} = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

where m_{tan} is the slope of the line tangent to $y=f(x)$ at $x=a$.

Ex. Find the slope of the line tangent to $y=x^2$ at $x=2$.

$$\lim_{h\to 0} \frac{(2+h)^2-2^2}{h} = \lim_{h\to 0} \frac{4+4h+h^2-4}{h}$$

$$= \lim_{h\to 0} \frac{4h+h^2}{h}$$

$$= \lim_{h\to 0} 4+h = 4.$$

So $\boxed{m_{tan} = 4.}$

Ex. Find m_{tan} for $f(x) = \frac{1}{x}$ at $x=3$.

$$\lim_{h\to 0} \frac{\frac{1}{3+h}-\frac{1}{3}}{h}$$ Note $f(3+h) = \frac{1}{3+h}$, not $\frac{1}{3}+h$

$$= \lim_{h\to 0} \frac{3-(3+h)}{3h(3+h)}$$

$$= \lim_{h\to 0} \frac{-h}{3h(3+h)}$$

$$= \lim_{h\to 0} -\frac{1}{3(3+h)} = -\frac{1}{9}$$ $$\boxed{m_{tan} = -1/9}$$

Ex. An automobile enters the turnpike at 10:15 a.m. and exits at 10:31 after traveling 20 miles. A week later the driver receives a speeding ticket. Why?

Let $s = f(t)$ be the position of the car at time t. If we start a clock when the car enters the turnpike then

$$S(0) = 0 \text{ miles}$$
$$S(16) = 20 \text{ miles}$$

Where t is measured in minutes. Since distance = rate x time we can compute

$$\text{rate} = \text{distance}/\text{time}$$

$$= 20 \text{ miles}/16 \text{ min}$$

$$= 1.25 \text{ miles per minute.}$$

Converting to miles per hour we find

$$1.25 \text{ miles}/\text{min} \times 60 \text{ min}/1 \text{ hr} = 75 \text{ mph}$$

The speed of the car was 75 miles per hour.

When? Was the speed constant?

Tangent Lines and Velocity

We computed the average speed of the car. If the average speed was 75 mph and at the beginning and end the car's speed was zero then at some point the car's speed must have been greater than 75 mph. (Consequence of the intermediate value theorem).

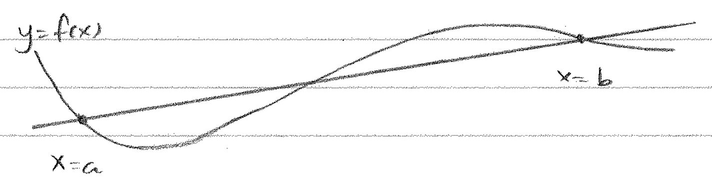
Often it is more helpful for us to talk about <u>velocity</u> rather speed. Knowing the velocity means you know not only the speed of the object, but what direction it is traveling in.

A negative velocity $v = -a$ means a speed of $|-a| = a$ but in the opposite direction.

If a particle is at $f(a)$ at time $t=a$ and at $f(b)$ at time $t=b$ then

The average velocity V_{avg} is $\dfrac{f(b)-f(a)}{b-a}$.

Note that if $b = a+h$ this is the difference quotient.



Slope of line is V_{avg}.

How can we find the velocity at a particular point? This is called the <u>instantaneous velocity</u> at $t=a$.

$$V(a) = \lim_{h \to 0} \frac{f(a+h) - f(h)}{h}$$, provided this limit exists.