The Derivative

Def. The <u>derivative</u> of the function $f(x)$ at $x=a$ is defined as

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h},$$

provided the limit exists. If the limit exists, we say that f is <u>differentiable</u> at $x=a$.

An alternative but equivalent definition is

$$f'(a) = \lim_{b\to a} \frac{f(b)-f(a)}{b-a}$$

since this reduces to the first form if $b=a+h$.

Ex Find the derivative of $f(x) = 4x^2+2x$ at $x=3$.

$$f'(3) = \lim_{h\to 0} \frac{f(3+h)-f(3)}{h}$$

$$= \lim_{h\to 0} \frac{4(3+h)^2+2(3+h)-[4\cdot 3^2+2\cdot 3]}{h}$$

$$= \lim_{h\to 0} \frac{4(9+6h+h^2)+2(3+h)-4\cdot 9-2\cdot 3}{h}$$

$$= \lim_{h\to 0} \frac{24h+4h^2+2h}{h}$$

$$= \lim_{h\to 0} \frac{h(26+4h)}{h} = \lim_{h\to 0} 26+4h = 26$$

$$\boxed{f'(3) = 26}$$

Ex. Find $f'(2)$ if $f(x) = 4x^2+2x$

We need to repeat the steps above.

Ex Find $f'(5)$ if $f(x) = 4x^2+2x$.

Repeat again

We may as well try to do all these calculations once in general.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{4(x+h)^2+2(x+h)-[4x^2+2x]}{h}$$

$$= \lim_{h\to 0} \frac{4(x^2+2xh+h^2)+2x+2h-4x^2-2x}{h}$$

$$= \lim_{h\to 0} \frac{4x^2+8xh+h^2+2x+2h-4x^2-2x}{h}$$

$$= \lim_{h\to 0} \frac{8xh+h^2+2h}{h}$$

$$= \lim_{h\to 0} 8x+h+2$$

$$= 8x+2$$

$$\boxed{f'(x) = 8x+2}$$

Now we can easily evaluate $f'(x)$ for any x.

Def The <u>derivative</u> of $f(x)$ is the function $f'(x)$ given by

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

provided the limit exists.

Ex Find $f'(x)$ if $f(x) = 1/x$.

$$f'(x) = \lim_{h\to 0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$= \lim_{h\to 0} \frac{\frac{x-(x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0} \frac{-h}{x(x+h)h}$$

$$= \lim_{h\to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$$

$$\boxed{f'(x) = -1/x^2}$$

Ex Find $f'(x)$ if $f(x) = \sqrt{x-1}$

First, notice that the domain of $f(x)$ is $x \geq 1$.

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{(x+h)-1} - \sqrt{x-1}}{h} \cdot \frac{\sqrt{(x+h)-1} + \sqrt{x-1}}{\sqrt{(x+h)-1} + \sqrt{x-1}}$$

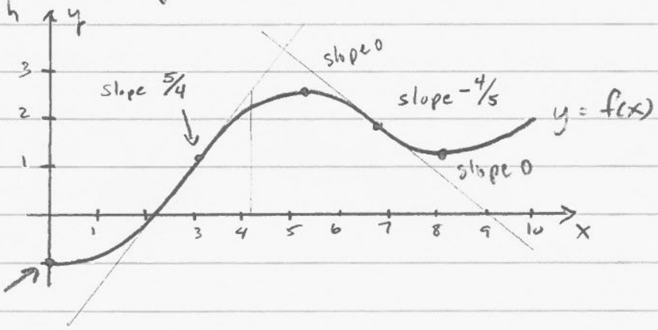
$$= \lim_{h \to 0} \frac{(x+h)-1-(x-1)}{h\left(\sqrt{(x+h)-1} + \sqrt{x-1}\right)}$$

$$= \lim_{h \to 0} \frac{h}{h\left(\sqrt{(x+h)-1} + \sqrt{x-1}\right)}$$

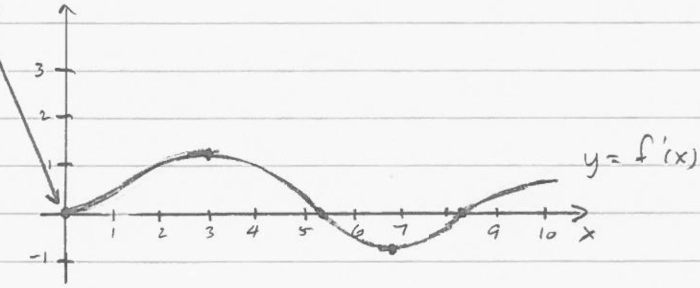
$$= \lim_{h \to 0} \frac{1}{\sqrt{(x+h)-1} + \sqrt{x-1}}$$

$$= \frac{1}{\sqrt{x-1} + \sqrt{x-1}} = \frac{1}{2\sqrt{x-1}} \qquad \therefore \boxed{f'(x) = \frac{1}{2\sqrt{x-1}}}$$

Ex Sketch the graph of $f'(x)$ if $f(x)$ is given in the graph



Slope zero gives zero



Notice: $f'(x) > 0$ where $f(x)$ has positive sloped tangents

$f'(x) < 0$ where $f(x)$ has negative sloped tangents

$f'(x) = 0$ where $f(x)$ has horizontal tangents.

Ex. Let $f(x) = \begin{cases} 2x & \text{if } x < 1 \\ x+1 & \text{if } x \geq 1 \end{cases}$

Notice that $f(x)$ is continuous since $2x$ and $x+1$ are both continuous and

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2x = 2$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x+1 = 2$$

So $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1) = 2$.

Find $f'(1)$ if possible.

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \begin{cases} \lim_{h\to 0^-} \frac{2(x+h)-2x}{h} & \text{or} \\ \lim_{h\to 0^+} \frac{(x+h)+1-(x+1)}{h} \end{cases}$$

$$\lim_{h\to 0^-} \frac{2(x+h)-2x}{h} = \lim_{h\to 0^-} \frac{2x+2h-2x}{h} = \lim_{h\to 0} \frac{2h}{h} = 2$$

$$\lim_{h\to 0^+} \frac{(x+h)+1-x-1}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

Since these limits do not agree, we conclude that $f'(1)$ does not exist.

Thm If $f(x)$ is differentiable at $x=a$ then $f(x)$ is continuous at $x=a$.

differentiability implies continuity, converse is not true.

Proof: We need to show $\lim_{x\to a} f(x) = f(a)$.

$$\lim_{x\to a} [f(x) - f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a)$$

$$= \left[\lim_{x\to a} \frac{f(x)-f(a)}{x-a}\right]\left[\lim_{x\to a}(x-a)\right]$$

$$= f'(a)\cdot 0 \qquad \text{if } f'(a) \text{ exists}$$

$$= 0.$$

$$\therefore \lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a} f(x) - \lim_{x\to a} f(a) = 0$$

so $\lim_{x\to a} f(x) = f(a)$ ✓

The Derivative

A word or two about notation...

We read $f'(x)$ as "f prime of x". This notation for the derivative is based on notation introduced by Newton

An alternative notion, introduced by Leibniz, is also commonly used.

$$\text{If } y = f(x) \text{ then } y' = \frac{dy}{dx} = \frac{d}{dx}f(x) = \frac{df}{dx}$$

This notation reminds us of $\Delta y/\Delta x$, change in y over change in x.

We read dy/dx as "d y over d x" or just "d y d x."

Suppose we want to write $f'(3)$ in Leibniz notation. We need to write

$$\left.\frac{df}{dx}\right|_{x=3}.$$

We will use these two notations interchangeably, taking advantage of the strengths of each of them.