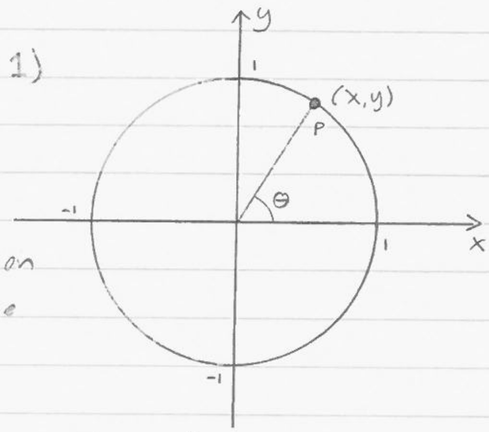
# Trigonometric and Inverse Trigonometric Functions

Consider a <u>unit circle</u> (circle of radius 1) centered at <u>the origin</u>.




We want to find the x and y coordinates of the point P on the circle as a function of the angle $\theta$ measured counterclockwise from the positive x-axis.

We're not the first to want to do this — but there is a problem — no such functions exist among the polynomials or rational functions.

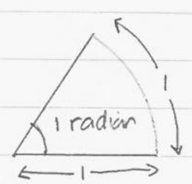
This problem is solved by definition. We define the functions <u>sine</u> and <u>cosine</u> to satisfy

$$x = \cos\theta \qquad \text{and} \qquad y = \sin\theta$$

* Note that $\sin\theta$ is really $\sin(\theta)$ and is not $\sin\cdot\theta$. This means $2\sin\theta \neq \sin 2\theta$ and $\sin\theta + \sin\phi \neq \sin(\theta+\phi)$.

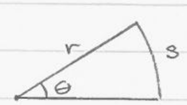
Question to ponder until next semester: How does your calculator compute sine and cosine?

For reasons that will become clearer later in the semester we will use <u>radians</u> to measure angles rather than <u>degrees</u>.




1 radian is defined to be the angle that subtends an arc of length 1 on the unit circle.

Although they may seem more complicated than degrees, radians actually will make many of our calculations much simpler.




$s = r\theta$

Circle circumference $= 2\pi r$ where $2\pi$ is the "angle"

# Trigonometric and Inverse Trigonometric Functions

Both sine and cosine are <u>periodic</u>.

Def: A function $f(x)$ is <u>periodic</u> of period $T$ if

$$f(x+T) = f(x)$$

for all $x$ for which $x$ and $x+T$ are in the domain of $f$.

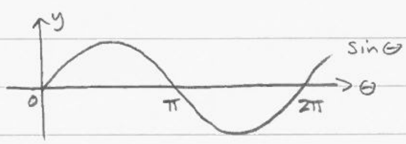
The smallest number $T>0$ is called the <u>fundamental period</u>.

<u>Thm</u> Both $\sin\theta$ and cosine $\theta$ are periodic with fundamental period $2\pi$.

Proof follows directly from definitions of sine, cosine, and radian measure.

Ex Solve $\sin 3x = 0$

From the definition we know $\sin\theta = 0$ when $\theta = 0$ and $\theta = \pi$.

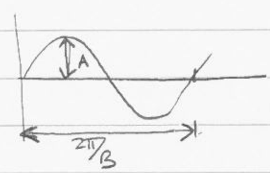



In fact, $\sin\theta = 0$ for $\theta = n\pi$ for all integers $n$.
This means $3x$ must be an integer multiple of $\pi$ so

$$3x = n\pi \implies x = \frac{n\pi}{3}.$$

$\therefore$ $\boxed{\sin 3x = 0 \text{ for any } x = \frac{n\pi}{3} \text{ where } n \text{ is an integer.}}$

<u>Ex</u> Consider $y = A\cos Bx$




$A$ is the <u>amplitude</u>

$\frac{B}{2\pi}$ is the <u>frequency</u>

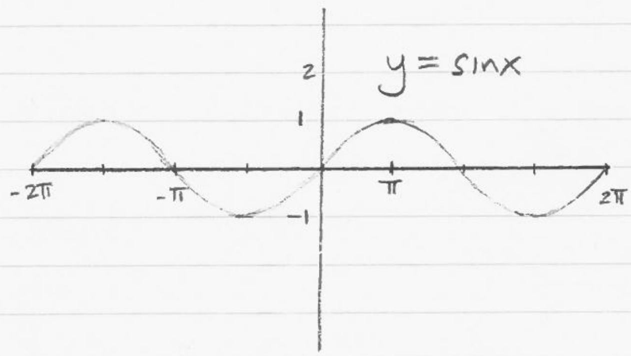
$\frac{2\pi}{B}$ is the <u>period</u>

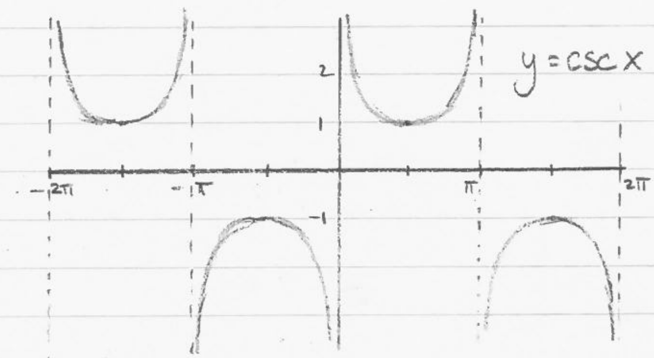
# Trigonometric and Inverse Trigonometric Functions

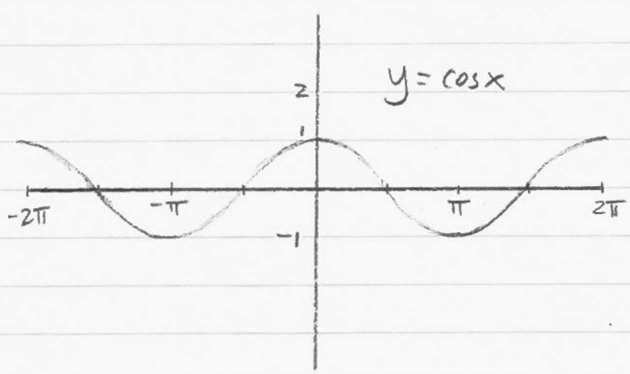


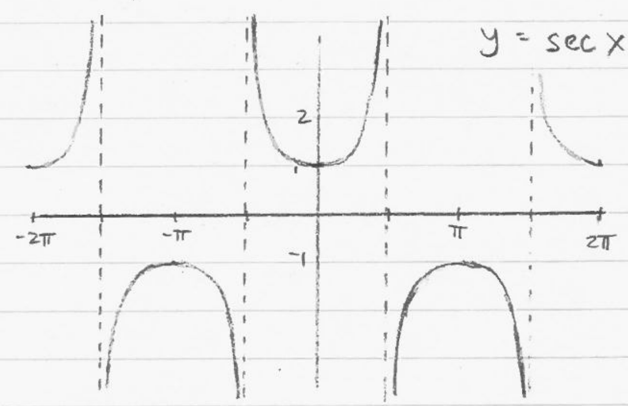
Other useful functions can be defined

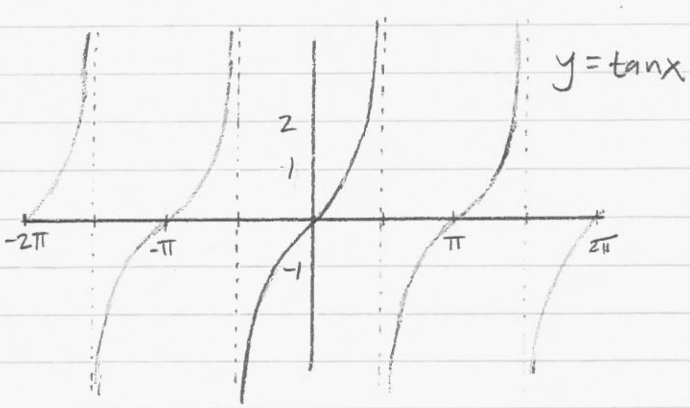
| | | |
|---|---|---|
| tangent | $\tan x =$ | $\sin x / \cos x$ |
| cotangent | $\cot x =$ | $\cos x / \sin x$ |
| secant | $\sec x =$ | $1/\cos x$ |
| cosecant | $\csc x =$ | $1/\sin x$ |

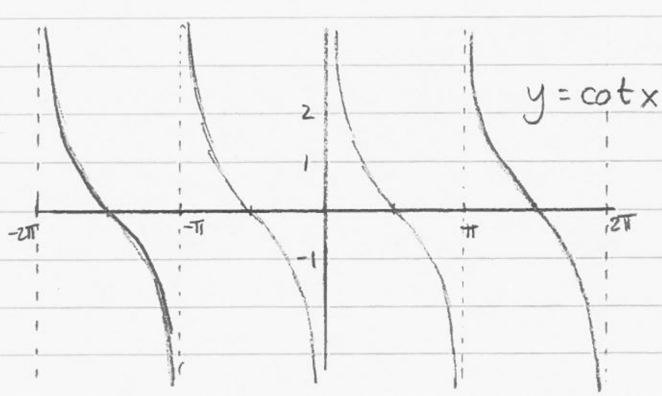


# Trigonometric and Inverse Trigonometric Functions

To define the inverse of the sine function we restrict the domain of $\sin x$ to $[-\pi/2, \pi/2]$ so $\sin x$ will be one-to-one. We then define

$$y = \sin^{-1} x \quad \text{if and only if} \quad x = \sin y$$

Notice that the range of $\sin^{-1} x$ is $[-\pi/2, \pi/2]$ and the domain is $[-1, 1]$.

We define the inverse cosine similarly, but the domain restriction is $[0, \pi]$.

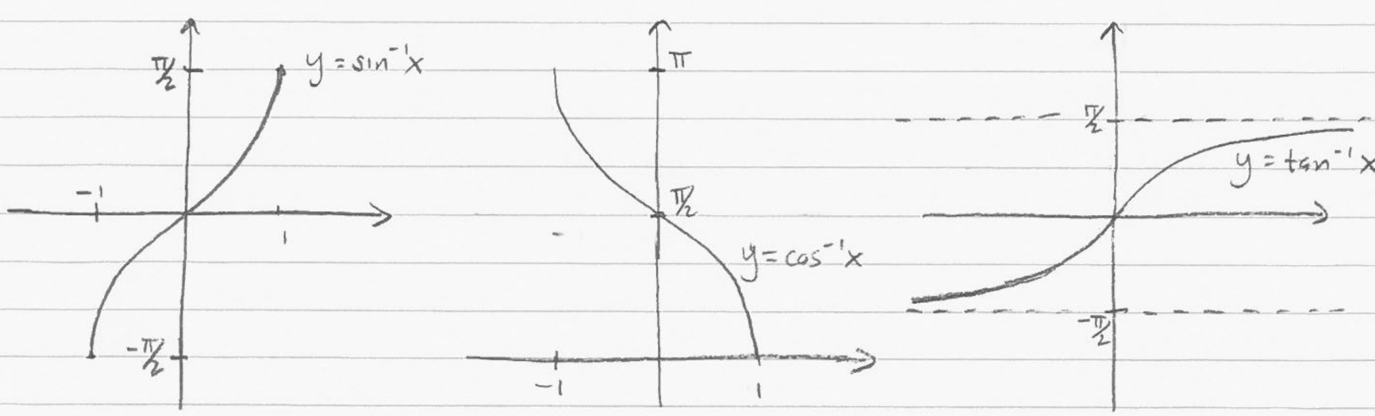
$$y = \cos^{-1} x \quad \text{iff} \quad x = \cos y$$

Domain: $[-1, 1]$
Range: $[0, \pi]$

For tangent we choose the restriction $(-\pi/2, \pi/2)$ — notice that the endpoints are not included — and define

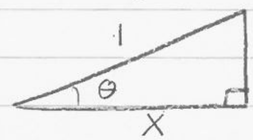
$$y = \tan^{-1} x \quad \text{iff} \quad x = \tan y$$

Domain: $(-\infty, \infty)$
Range: $(-\pi/2, \pi/2)$

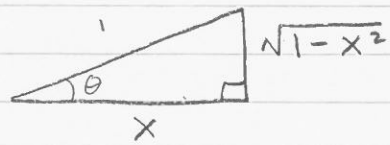


# Trigonometric and Inverse Trigonometric Functions

Ex Simplify $\tan(\cos^{-1}x)$ Let $\theta = \cos^{-1}x$




$\cos\theta = x = \frac{x}{1}$




$\tan(\cos^{-1}x) = \tan\theta = \frac{\sqrt{1-x^2}}{x}$

$$\boxed{\tan(\cos^{-1}x) = \frac{\sqrt{1-x^2}}{x}}$$

Note: $\sin^{-1}x$ is also denoted $\arcsin x$ which, although longer, may be preferable to avoid confusion with the reciprocal of $\sin x$ (i.e. $\csc x$).