Parallel Sorting

CPS343

Parallel and High Performance Computing

Spring 2020

Outline

1. Overview of Sorting
   - Sorting Characteristics
   - Bubble Sort
   - Quicksort
   - Mergesort
2. Parallel Sorting
   - Assumptions
   - Parallel Quicksort
   - Parallel Sample Sort
   - Parallel Sorting by Regular Sampling (PSRS)

Outline

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\begin{aligned}
&\text{if } a_i > a_j \text{ then} \\
&\qquad t = a_i \\
&\qquad a_i = a_j \\
&\qquad a_j = t \\
&\text{endif}
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$a_i = a_j$
$a_j = t$
endif

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- An example that uses this approach is the bubble sort
- Other sorts (e.g. mergesort) do not have compare and exchange as a basic operation.

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- Given an n element list of integers a_i, $i = 0, 1, \ldots, n-1$:

```
done = false
last = n − 2
while not done do
    done = true
    for i = 0 to last do
        if a_i > a_{i+1} then
            done = false
            swap(a_i, a_{i+1})
        endif
    endfor
    last = last − 1
endwhile
```

Bubble Sort

For example, consider the bubble sort applied to a list of eight numbers. The number in **bold** in each column represents a_i and is compared with the next number in the list.

5
7
2
8
4
1
3
6

Bubble Sort

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5	5
7	**7**
2	2
8	8
4	4
1	1
3	3
6	6

Bubble Sort

For example, consider the bubble sort applied to a list of eight numbers. The number in **bold** in each column represents a_i and is compared with the next number in the list.

5	5	5
7	**7**	2
2	2	**7**
8	8	8
4	4	4
1	1	1
3	3	3
6	6	6

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5	5	5	5
7	**7**	2	2
2	2	**7**	7
8	8	8	**8**
4	4	4	4
1	1	1	1
3	3	3	3
6	6	6	6

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5	5	5	5	5
7	**7**	2	2	2
2	2	**7**	7	7
8	8	8	**8**	4
4	4	4	4	**8**
1	1	1	1	1
3	3	3	3	3
6	6	6	6	6

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7	**7**	2	2	2	2
2	2	**7**	7	7	7
8	8	8	**8**	4	4
4	4	4	4	**8**	1
1	1	1	1	1	**8**
3	3	3	3	3	3
6	6	6	6	6	6

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7	**7**	2	2	2	2	2
2	2	**7**	7	7	7	7
8	8	8	**8**	4	4	4
4	4	4	4	**8**	1	1
1	1	1	1	1	**8**	3
3	3	3	3	3	3	**8**
6	6	6	6	6	6	6

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7	**7**	2	2	2	2	2	2
2	2	**7**	7	7	7	7	7
8	8	8	**8**	4	4	4	4
4	4	4	4	**8**	1	1	1
1	1	1	1	1	**8**	3	3
3	3	3	3	3	3	**8**	6
6	6	6	6	6	6	6	**8**

Bubble Sort

The second sweep looks like

5
2
7
4
1
3
6
8

Bubble Sort

The second sweep looks like

5	2
2	**5**
7	7
4	4
1	1
3	3
6	6
8	8

Bubble Sort

The second sweep looks like

5	2	2
2	**5**	5
7	7	**7**
4	4	4
1	1	1
3	3	3
6	6	6
8	8	8

Bubble Sort

The second sweep looks like

5	2	2	2
2	**5**	5	5
7	7	**7**	4
4	4	4	**7**
1	1	1	1
3	3	3	3
6	6	6	6
8	8	8	8

Bubble Sort

The second sweep looks like

5	2	2	2	2
2	**5**	5	5	5
7	7	**7**	4	4
4	4	4	**7**	1
1	1	1	1	**7**
3	3	3	3	3
6	6	6	6	6
8	8	8	8	8

Bubble Sort

The second sweep looks like

5	2	2	2	2	2
2	**5**	5	5	5	5
7	7	**7**	4	4	4
4	4	4	**7**	1	1
1	1	1	1	**7**	3
3	3	3	3	3	**7**
6	6	6	6	6	6
8	8	8	8	8	8

Bubble Sort

The second sweep looks like

5	2	2	2	2	2	2
2	**5**	5	5	5	5	5
7	7	**7**	4	4	4	4
4	4	4	**7**	1	1	1
1	1	1	1	**7**	3	3
3	3	3	3	3	**7**	6
6	6	6	6	6	6	**7**
8	8	8	8	8	8	8

Bubble Sort

The second sweep looks like

5	2	2	2	2	2	2
2	**5**	5	5	5	5	5
7	7	**7**	4	4	4	4
4	4	4	**7**	1	1	1
1	1	1	1	**7**	3	3
3	3	3	3	3	**7**	6
6	6	6	6	6	6	**7**
8	8	8	8	8	8	8

Notice that we have one fewer comparison since the largest element in the list is already at the bottom.

Bubble Sort

The third sweep looks like

2
5
4
1
3
6
7
8

Bubble Sort

The third sweep looks like

2	2
5	**5**
4	4
1	1
3	3
6	6
7	7
8	8

Bubble Sort

The third sweep looks like

2	2	2
5	**5**	4
4	4	**5**
1	1	1
3	3	3
6	6	6
7	7	7
8	8	8

Bubble Sort

The third sweep looks like

2	2	2	2
5	**5**	4	4
4	4	**5**	1
1	1	1	**5**
3	3	3	3
6	6	6	6
7	7	7	7
8	8	8	8

Bubble Sort

The third sweep looks like

2	2	2	2	2
5	**5**	4	4	4
4	4	**5**	1	1
1	1	1	**5**	3
3	3	3	3	**5**
6	6	6	6	6
7	7	7	7	7
8	8	8	8	8

Bubble Sort

The third sweep looks like

2	2	2	2	2	2
5	**5**	4	4	4	4
4	4	**5**	1	1	1
1	1	1	**5**	3	3
3	3	3	3	**5**	5
6	6	6	6	6	**6**
7	7	7	7	7	7
8	8	8	8	8	8

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2	2	2	2	2	2
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4	4	**5**	1	1	1
1	1	1	**5**	3	3
3	3	3	3	**5**	5
6	6	6	6	6	**6**
7	7	7	7	7	7
8	8	8	8	8	8

Now the last three elements are all in place. After four more sweeps the list will be sorted.

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- This means that in the worst case the number comparisons is $(n-1)+(n-2)+\cdots+2+1 = n(n-1)/2$ which is $O(n^2)$.
- The number of comparisons in the average case is also $O(n^2)$.

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- Quicksort is naturally recursive.
- Average number of comparisons is $O(n \log n)$.
- Worst case number of comparisons is $O(n^2)$.

Quicksort

Often the first element in the list is used as the pivot, although other mechanisms can be used. Ideally pivot is median of sorted list.

Here red indicates the selected pivot and elements in blue are previous pivots that now partition the list.

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7
2
8
4
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3
6

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7	4
2	1
8	3
4	**5**
1	7
3	8
6	6

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8	3	3	3	3
4	**5**	**5**	**5**	**5**
1	7	**7**	6	6
3	8	8	**7**	**7**
6	6	6	8	8

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2	1	1	4	**4**	3
8	3	3	3	3	**4**
4	**5**	**5**	**5**	**5**	**5**
1	7	**7**	6	6	6
3	8	8	**7**	**7**	**7**
6	6	6	8	8	8

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7	4	4	**2**	**2**	**2**	2
2	1	1	4	**4**	3	3
8	3	3	3	3	**4**	4
4	**5**	**5**	**5**	**5**	**5**	5
1	7	**7**	6	6	6	6
3	8	8	**7**	**7**	**7**	7
6	6	6	8	8	8	8

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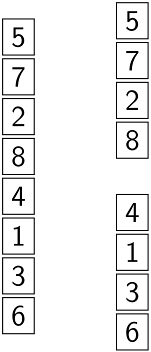
- Basic algorithm:
  1. if list length is 0 or 1 then return.
  2. partition list into two sublists of (nearly) equal size.
  3. sort each sublist using mergesort.
  4. merge both sublists into one sorted list.

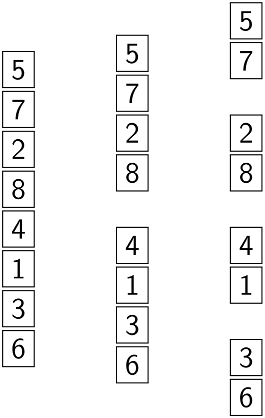
Mergesort

$\boxed{5}$
$\boxed{7}$
$\boxed{2}$
$\boxed{8}$
$\boxed{4}$
$\boxed{1}$
$\boxed{3}$
$\boxed{6}$

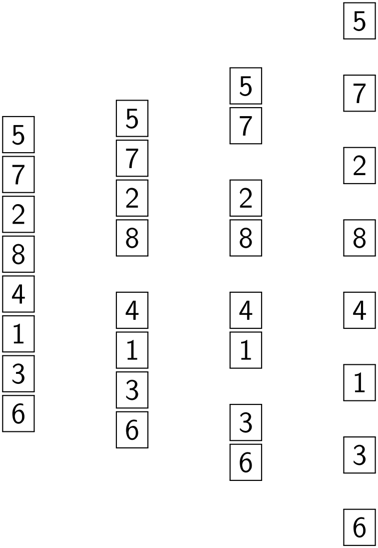
Mergesort

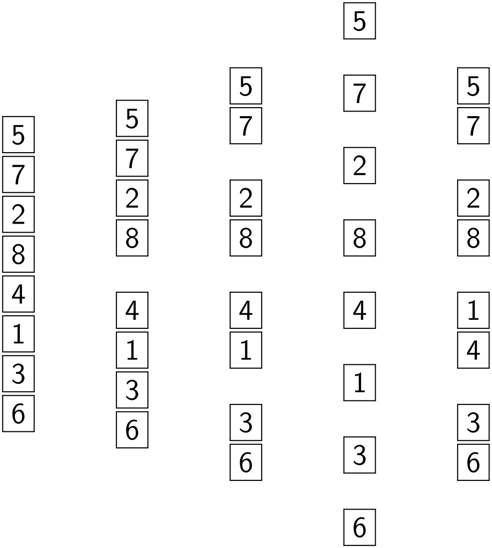


Mergesort



Mergesort

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5 7 2 8
4 1 3 6
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4 1
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7
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1
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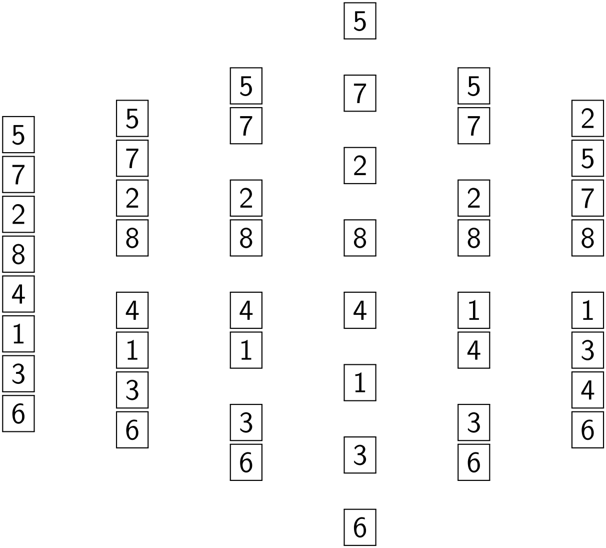
6
5 7
2 8
1 4
3 6

Mergesort



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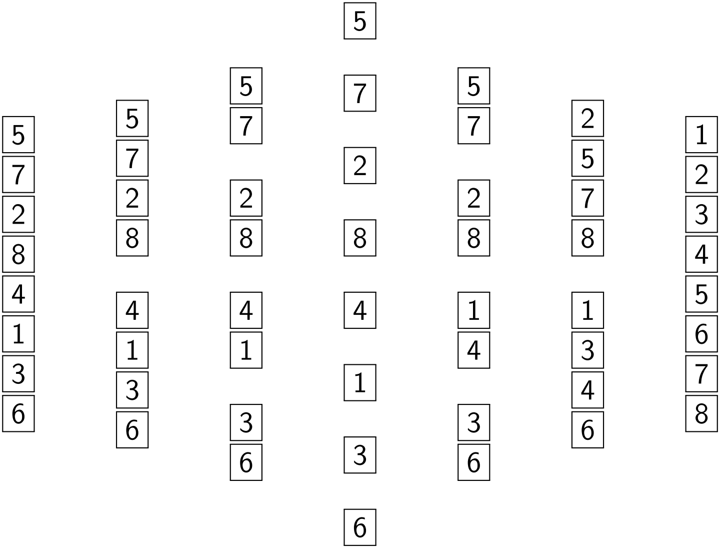
5 7 2 8 4 1 3 6
5 7 2 8 | 4 1 3 6
5 7 | 2 8 | 4 1 | 3 6
5 | 7 | 2 | 8 | 4 | 1 | 3 | 6
5 7 | 2 8 | 1 4 | 3 6
2 5 7 8 | 1 3 4 6
1 2 3 4 5 6 7 8

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General Assumptions

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- computer is a parallel cluster with p nodes.

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- computer is a parallel cluster with p nodes.
- list of length n is stored such that each cluster node has a approximately $\lceil n/p \rceil$ elements of the list.
- sort keys are integers

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- Assume number of processes p and total length of the list n are both powers of 2.

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Parallel Quicksort Algorithm Assumptions

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- *Partner* processes have the same ID except for the leading bit.
- For example, suppose there are eight processes. Then we'd have

lower half	upper half
000	100
001	101
010	110
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- The upper and lower halves of the process list are now treated as separate process lists each with $p/2$ processes. Appropriate upper and lower half lists are identified and the algorithm recurses.

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- After $\log_2 p$ recursions each process has its final portion of the entire list; all that remains is to sort it using a sequential sort.

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- Unfortunately, the sublist sizes in each process are unlikely to remain uniform. This means that the comparison work will become unbalanced and, in the worst case where one process is responsible for a majority of the list, approach the level for a sequential algorithm.
- Of course, once the list elements have all been collected in the proper processes, they must still be sorted using a sequential sort.

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  (This assumes that send/receive operations are not overlapped.)
- Summing these two costs we find the communication cost per step is

$$t_{\text{comm per step}} = t_{\text{broadcast}} + t_{\text{exchange}} = 3t_{\text{startup}} + \left(1 + \frac{n}{p}\right)t_{\text{data}}.$$

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- Since there are $\log_2 p$ steps, the total communication cost for all steps is
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- As already mentioned, however, the average sublist size in each process can vary dramatically as the algorithm proceeds, meaning this analysis may not apply to the typical case.

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  1. Each process sorts its original portion of the list.
  2. Splitters are chosen and distributed.
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- The splitters are then broadcast to all processes.
- One potential complication is that splitters must be unique; elements in the original list can be "tagged" in some way so that list can be uniquely ordered.

Sample Sort Example

- Original list distributed on two processors ($p = 2$)

12	14	8	16	2	13	4	11

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- Samples are then sorted

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Sample Sort Example (continued)

The sorted samples: $\boxed{\boxed{7}\ \boxed{8}\ \boxed{10}\ \boxed{14}}$

- Since there are only two processes, there is only one splitter, which is located at position $1s$. As $s = 2$, the single element splitter list contains only the element 10.

Sample Sort Example (continued)

The sorted samples: | 7 | 8 | 10 | 14 |

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- Finally, each process sequentially sorts its portion of the list.

1	2	3	4	5	6	7	8	9

10	11	12	13	14	15	16

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- As our example illustrates, it is often the case that the sorting operation will end with an imbalance in distribution of the sorted list. Usually this will be smaller than with the parallel Quicksort.
- The number of splitters s can impact the performance of the algorithm; in general smaller s values lead to greater load imbalances. As s increases the sequential work to determine the splitters increases.
- During the data exchange, processes must be prepared to accept an unknown amount of data. To accommodate this we can use an additional global communication (e.g. using `MPI_Allgather()`) to distribute the partition sizes each process is preparing to send.

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Parallel Sorting by Regular Sampling (PSRS)

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- Rather than using a random sample to determine the splitters, the PSRS selects specific elements of the locally sorted list as samples.
- The phrase "regular sampling" implies that the elements are selected at regular intervals in the local list.
- Since each locally sorted list has roughly n/p elements, choosing p regularly spaced samples can be done by selecting samples with indices

$$0, \left(\frac{1}{p} \cdot \frac{n}{p}\right), \left(\frac{2}{p} \cdot \frac{n}{p}\right), \ldots, \left(\frac{p-1}{p} \cdot \frac{n}{p}\right)$$

which reduces to

$$0, \frac{n}{p^2}, \frac{2n}{p^2}, \ldots, \frac{(p-1)n}{p^2}.$$